MOTIVIC RENORMALIZATION AND SINGULARITIES

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Così tra questa
infinità s'annega il pensier mio:
e 'l naufragar m'è dolce in questo mare.
(Giacomo Leopardi, *L'infinito*, from the second handwritten version)

To Alain Connes, on his 60th birthday and many other occasions

Abstract. We consider parametric Feynman integrals and their dimensional regularization from the point of view of differential forms on hypersurface complements and the approach to mixed Hodge structures via oscillatory integrals. We consider restrictions to linear subspaces that slice the singular locus, to handle the presence of non-isolated singularities. In order to account for all possible choices of slicing, we encode this extra datum as an enrichment of the Hopf algebra of Feynman graphs. We introduce a new regularization method for parametric Feynman integrals, which is based on Leray coboundaries and, like dimensional regularization, replaces a divergent integral with a Laurent series in a complex parameter. The Connes–Kreimer formulation of renormalization can be applied to this regularization method. We relate the dimensional regularization of the Feynman integral to the Mellin transforms of certain Gelfand–Leray forms and we show that, upon varying the external momenta, the Feynman integrals for a given graph span a family of subspaces in the cohomological Milnor fibration. We show how to pass from regular singular Picard–Fuchs equations to irregular singular flat equisingular connections. In the last section, which is more speculative in nature, we propose a geometric model for dimensional regularization in terms of logarithmic motives and motivic sheaves.

Contents

1. Introduction

We consider here perturbative quantum field theories governed by a Lagrangian, which in a Lorentzian metric of signature $(+1, -1, \dots, -1)$ on the flat D-dimensional spacetime $\mathbb{R}^D$, is given in the form

$$\mathcal{L}(\phi) = \frac{1}{2}(\partial\phi)^2 - \frac{m^2}{2}\phi^2 - \mathcal{L}_{int}(\phi), \tag{1.1}$$

where the interaction term $\mathcal{L}_{int}(\phi)$ is polynomial in ϕ of degree at least three. The corresponding action functional $S(\phi) = \int \mathcal{L}(\phi) d^D x$ involves a single scalar field ϕ. This is the simplest case, considered in the work of Connes–Kreimer. Generalizations of the Connes–Kreimer formalism for other theories have been developed more recently (see for instance [41] for the case of gauge theories), but for the purposes of the present paper we restrict our attention to scalar theories.

The purpose of this paper is to relate the approach to renormalization of Connes–Kreimer [21], via Birkhoff factorization of loops in the Lie groups of characters of the Hopf algebra of Feynman graphs, and its successive reformulation of Connes–Marcolli [22] in terms of Galois theory of a category of flat equisingular connections with irregular singularities, to the approach via parametric Feynman integrals, periods of complements of graph hypersurfaces, and motives, developed by Bloch–Esnault–Kreimer in [14], [13].

The main approach we follow in this paper, in order to bridge between these two different approaches is a formulation of the dimensionally regularized Feynman integrals in terms of Mellin transforms of certain Gelfand–Leray forms, as in the approach of Varchenko [43], [44] to the theory of singularities and asymptotic mixed Hodge structures on the cohomological Milnor fibration, in terms of asymptotic properties of oscillatory integrals.

We deal with the fact that the graph hypersurfaces tend to have non-isolated singularities by slicing the Feynman integral along generic linear spaces of dimension at most equal to the codimension of the singular locus, using the same kind of techniques used in the integral geometry of Radon transforms in projective spaces developed by Gelfand–Gindikin–Graev [29]. Since typically the singular locus is rather large in dimension, the slices obtained in this way will often be singular curves in $\mathbb{P}^2$ or singular surfaces in $\mathbb{P}^3$. Instead of considering a single choice of a slicing, which would mean losing too much information on the graph hypersurface, one considers all possible choices and implements the data of the cutting linear space as part of the Hopf algebra of graphs, much like what one does with the choice of the external momenta, so that all possible choices are considered as part of the structure.

The formulation one obtains in this way, in terms of Gelfand–Leray forms, suggests a new method of regularization of parametric Feynman integrals, which, as in the case of dimensional regularization, replaces a divergent integral with a Laurent series in a complex variable ϵ, but which is defined using Leray coboundaries to avoid the singular locus, by integrating around it along the fibers of a circle bundle. We check that the formulation of renormalization in terms of Hopf algebras and Birkhoff factorization developed in Connes–Kreimer [21] applies without changes if one uses this new regularization method instead of the customary dimensional regularization.

The interpretation of the dimensionally regularized Feynman integrals as Mellin transforms of Gelfand–Leray forms provides a direct link between Feynman integrals and the cohomological Milnor fibration. In particular, we prove that, upon varying the external momenta and the spacetime dimension $D \in \mathbb{N}$ in which the scalar theory is considered, the corresponding Feynman integrals determine a family of subspaces of the cohomological Milnor fibration, which inherit a Hodge and a weight filtration from the asymptotic mixed Hodge structure of Varchenko. It remains to be seen when this subspace recovers the full Milnor fiber cohomology and/or when these induced filtrations still define a mixed Hodge structure.

Another important question, in trying to compare the approaches of [22] and [14] is the use of irregular, as opposed to regular singular, connections. In fact, from the point of view of motives or mixed Hodge structures, what one expects to find is regular singular connections. These appear naturally in the form of Picard–Fuchs equations and Gauss–Manin connections. However, the Galois theory approach to the classification of divergences in perturbative quantum field theory developed in [22] relies on the use of irregular singular connections and a form of the Riemann–Hilbert correspondence based on Ramis' wild fundamental group. We reconcile these two approaches by showing that, upon passing to Mellin transforms of solutions of a regular singular Picard–Fuchs equation, one obtains solutions of differential equations with irregular singularities. More precisely, we first recall the construction and properties of the irregular singular connections considered in [22] and the equisingularity condition that characterizes them. We then prove that solutions of the regular singular Picard–Fuchs equations at the singular points of a graph hypersurface (sliced with a linear space of a suitable dimension so that singularities are isolated) can be assembled to give rise to a solution of a differential system of the type considered in [22], with irregular singularities and with coefficients in the Lie algebra of the affine group scheme of the Hopf algebra of Feynman graphs of the theory, suitably enriched to account for the choice of the slicing of the Feynman integrals by linear spaces of the appropriate dimension.

Finally, we propose a motivic interpretation for dimensional regularization, in terms of the logarithmic extensions of Tate motives (the Kummer motives), and their pullbacks via the polynomial function defining the graph hypersurface. This amounts to associating to the Feynman graphs of a given scalar theory a subcategory of the Arapura category of motivic sheaves of [3]. We expect that this may provide a way of interpreting the relation between dimensionally regularized Feynman integrals and cohomological Milnor fibrations in terms of a motivic version of the Milnor fiber. We hope to relate, in this way, a motivic zeta function associated to the resulting mixed motive with the dimensionally regularized Feynman integral.

An in depth study of parametric Feynman integrals in perturbative renormalization and their relation to mixed Hodge structures was carried out in very recent work of Bloch and Kreimer [15].

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2. Parametric Feynman integrals

In this section we recall the Feynman parametric formulation of the momentum integrals associated to the Feynman graphs in the perturbative expansion of a scalar field theory. We also recall the Dimensional Regularization method and the form of the regularized integrals. These are all well known techniques, but we review them briefly for completeness. We also recall the explicit form of the graph polynomials $\Psi_\Gamma(t)$ and $P_\Gamma(t,p)$ and their properties, as well as the explicit mass scale dependence of the dimensionally regularized Feynman integrals. Moreover, in §2.4 we give a reformulation of the Feynman integrals in terms of differential forms on hypersurface complements in projective spaces.

2.1. **Feynman parameters and algebraic varieties.** We recall briefly the method for the computation of Feynman integrals based on the *parametric representation*. This is well known material in the physics literature, see *e.g.* §6-2-3 of [32], §18 of [11], and §6 of [38]. However, since it is not part of the standard mathematician's toolbox, we prefer to spend a few words here recalling the basic ideas.

The terms in the formal asymptotic expansion of functional integrals

$$\int \mathcal{O}(\phi) e^{\frac{i}{\hbar}S(\phi)} \mathcal{D}[\phi],$$

obtained by treating the interaction terms $S_{int}(\phi) = \int \mathcal{L}_{int}(\phi) d^D x$ as a perturbation, are labeled by Feynman graphs of the theory. The topology of these graphs is constrained by the requirement that the valence of each vertex is equal to the degree of one of the monomials in the Lagrangian. The edges of the graph are divided into internal lines, each connecting two vertices, and external lines, which are half-lines with one end attached to a vertex of the graph and one open end. The order in the expansion is given by the loop number of the graph, or by the number of internal lines. Each external line carries a datum of an external momentum $p \in \mathbb{R}^D$ with a conservation law

$$\sum_{e \in E_{ext}(\Gamma)} p_e = 0, \tag{2.1}$$

where $E_{ext}(\Gamma)$ is the set of external edges of Γ.

We assume that all our graphs are one-particle-irreducible (1PI), *i.e.* that they cannot be disconnected by cutting a single internal edge.

The Feynman rules assign to a Feynman graph a function $U(\Gamma) = U(\Gamma, p_1 \ldots, p_N)$ of the external momenta obtained by integrating, over momentum variables k_e assigned to each internal edge of Γ, an expression involving propagators for each internal line and momentum conservations at each vertex, in the form

$$U(\Gamma) = \prod_{v \in V(\Gamma)} (2\pi)^D \lambda_v \int \frac{\delta(\sum_{i=1}^n \epsilon_{v,i} k_i + \sum_{j=1}^N \epsilon_{v,j} p_j)}{q_1 \cdots q_n} \, d^D k_1 \cdots d^D k_n, \tag{2.2}$$

where $n = \# E_{int}(\Gamma)$, the number of internal edges in the graph, $N = \# E_{ext}(\Gamma)$, while λ_v are the coupling constants (the coefficients of the term in the Lagrangian of degree equal to the valence of the vertex $v \in V(\Gamma)$), and $\epsilon_{v,e}$ is the incidence matrix

$$\epsilon_{v,e} = \begin{cases} +1 & t(e) = v \\ -1 & s(e) = v \\ 0 & \text{otherwise,} \end{cases} \tag{2.3}$$

with $s(e)$ and $t(v)$ the source and target vertices of the oriented edge e. In the following we drop the multiplicative constants $(2\pi)^D\lambda_v$ and we concentrate on the remaining integral, which we still denote by $U(\Gamma)$.

The basic formula (2.2) is dictated by the Feynman rules of the given quantum field theory, which prescribe that the contribution of a Feynman graph to the perturbative expansion of the effective action is obtained as a product of an inverse propagator q_e^{-1} for each edge e, in a corresponding momentum variable, with linear relations among the momentum variables given by imposing conservation laws at each vertex that ensure momentum conservation, and with each vertex contributing a multiplicative factor depending on coupling constants and powers of (2π). These Feynman rules are modeled on similar series expansions of finite dimensional Gaussian integrals.

The q_i, for $i = 1, \dots, n$ are the quadratic forms defining the free field propagator associated to the corresponding line in the graph, namely

$$q_i(k) = k_i^2 - m^2 + i\epsilon \quad \text{or} \quad q_i(k) = k_i^2 + m^2, \tag{2.4}$$

respectively in the Lorentzian and in the Eucidean signature case. In the following, we work preferably in the Euclidean setting.

We refer to $U(\Gamma)$ as the *unrenormalized Feynman integral*. The parametric form of $U(\Gamma)$ is obtained by first introducing the *Schwinger parameters*, using the identity

$$\frac{1}{q} = \int_0^\infty e^{-sq} ds.$$

This gives the expression

$$\frac{1}{q_1 \cdots q_n} = \int_0^\infty \cdots \int_0^\infty e^{-(s_1 q_1 + \cdots + s_n q_n)} \, ds_1 \cdots ds_n, \tag{2.5}$$

which is a special case of the more general identity

$$\frac{1}{q_1^{k_1} \cdots q_n^{k_n}} = \frac{1}{\Gamma(k_1) \cdots \Gamma(k_n)} \int_0^\infty \cdots \int_0^\infty e^{-(s_1 q_1 + \cdots + s_n q_n)} \, s_1^{k_1 - 1} \cdots s_n^{k_n - 1} \, ds_1 \cdots ds_n. \tag{2.6}$$

The Feynman parametric form is obtained from this expression by a change of variables that replaces the Schwinger parameters $s_i \in \mathbb{R}_+$ with new variable $t_i \in [0, 1]$, by setting $s_i = St_i$ with $S = s_1 + \cdots + s_n$. This gives

$$\frac{1}{q_1^{k_1} \cdots q_n^{k_n}} = \frac{\Gamma(k_1 + \cdots + k_n)}{\Gamma(k_1) \cdots \Gamma(k_n)} \int_0^1 \cdots \int_0^1 \frac{t_1^{k_1 - 1} \cdots t_n^{k_n - 1} \, \delta(1 - \sum_{i=1}^n t_i)}{(t_1 q_1 + \cdots + t_n q_n)^{k_1 + \cdots + k_n}} \, dt_1 \cdots dt_n, \tag{2.7}$$

hence in particular one obtains

$$\frac{1}{q_1 \cdots q_n} = (n-1)! \int \frac{\delta(1 - \sum_{i=1}^n t_i)}{(t_1 q_1 + \cdots + t_n q_n)^n} \, dt_1 \cdots dt_n, \tag{2.8}$$

as an integration in the Feynman parameters $t = (t_i)$ over the simplex

$$\Sigma = \{t = (t_i) \in \mathbb{R}_+^n \mid \sum_i t_i = 1\}. \tag{2.9}$$

Next one introduces a further change of variables involving another matrix naturally associated to the graph, the circuit matrix η_{ik}, defined in terms of an orientation of the edges $e_i \in E(\Gamma)$ and a choice of a basis for the first homology group, $l_k \in H_1(\Gamma, \mathbb{Z})$, with $k = 1, \dots, \ell = b_1(\Gamma)$, by setting

$$\eta_{ik} = \begin{cases} +1 & \text{edge } e_i \in \text{loop } l_k, \text{ same orientation} \\ -1 & \text{edge } e_i \in \text{loop } l_k, \text{ reverse orientation} \\ 0 & \text{otherwise.} \end{cases} \tag{2.10}$$

We also define $M_\Gamma(t)$ to be the matrix

$$(M_\Gamma)_{kr}(t) = \sum_{i=0}^{n} t_i \eta_{ik}\eta_{ir}. \tag{2.11}$$

Notice that, while the matrix $M_\Gamma(t)$ depends on the choice of the orientation of the edges and of the choice of a basis for the first homology of Γ, the determinant $\det(M_\Gamma(t))$ is independent of both choices.

One then makes a change of variables in the quadratic forms q_i of (2.4), by setting

$$k_i = u_i + \sum_{k=1}^{\ell} \eta_{ik} x_k, \tag{2.12}$$

with the constraint

$$\sum_{i=0}^{n} t_i u_i \eta_{ik} = 0, \tag{2.13}$$

for all $k = 1, \dots, \ell$. The momentum conservation relations

$$\sum_{i=1}^{n} \epsilon_{v,i} k_i + \sum_{j=1}^{N} \epsilon_{v,j} p_j = 0$$

of (2.2) shows that the u_i in (2.12) also satisfy

$$\sum_{i=1}^{n} \epsilon_{v,i} u_i + \sum_{j=1}^{N} \epsilon_{v,j} p_j = 0. \tag{2.14}$$

This uses the fact that the incidence matrix $\epsilon = (\epsilon_{v,e})$ and the circuit matrix $\eta = (\eta_{e,k})$ satisfy $\epsilon\eta = \sum_{e\in E(\Gamma)} \epsilon_{v,e}\eta_{e,k} = 0$, *cf.* [38], §3. The two equations (2.13) and (2.14) are the analog for momenta in Feynman graphs of the Kirchhoff laws of circuits, respectively giving the conservation laws for the sum of voltage drops along a loop in a circuit and of incoming currents at a vertex, with momenta replacing currents and the Feynman parameters in the role of resistances ([11], §18).

The u_i are determined by (2.13) and (2.14), and one can write the term $\sum_i t_i(u_i^2 + m^2)$ in the form of a function of the Feynman parameters t and the external momenta p of the form

$$V_\Gamma(t,p) := p^\tau R_\Gamma(t) p + m^2, \tag{2.15}$$

where we use the fact that $\sum_i t_i = 1$. The $N \times N$-matrix $R(t)$, with $N = \#E_{ext}(\Gamma)$ is constructed out of another matrix associated to the graph. This is obtained as follows (*cf.* [32], §6-2-3). Let $D_\Gamma(t)$ denote the matrix

$$(D_\Gamma(t))_{v,v'} = \sum_{i=1}^{n} \epsilon_{v,i}\, \epsilon_{v',i}\, t_i^{-1}, \tag{2.16}$$

with $n = \#E_{int}(\Gamma)$ and with $\epsilon_{v,i}$ the incidence matrix as in (2.3). Then the quadratic form $p^\tau R(t) p$ of (2.15) has the form

$$p^\tau R_\Gamma p = \sum_{v,v'} P_v (D_\Gamma(t)^{-1})_{v,v'} P_{v'}, \tag{2.17}$$

where

$$P_v = \sum_{e\in E_{ext}(\Gamma), t(e)=v} p_e \tag{2.18}$$

is the sum of the incoming external momenta at the vertex v.

Summarizing the previous discussion, the result of the change of variables (2.12) is that we can rewrite the original Feynman integral (2.2) in the following form.

Lemma 2.1. *For $n - D\ell/2 > 0$, the Feynman integral* (2.2) *can be written, after the change of variables* (2.12)*, in the form*

$$\int_\Sigma \frac{\delta(1-\sum_i t_i)}{\det(M_\Gamma(t))^{D/2} V_\Gamma(t,p)^{n-D\ell/2}}\, dt_1\cdots dt_n, \tag{2.19}$$

up to a multiplicative constant.

Proof. This follows [11], §18 and [32] p.376. First recall the well known identity for the Gaussian integral

$$\int e^{-\frac{1}{2}x^\tau A x} d^D x_1 \cdots d^D x_\ell = \frac{(2\pi)^{D\ell/2}}{\det(A)^{D/2}}, \tag{2.20}$$

for A an $\ell\times\ell$ real symmetric matrix. We then have

$$\frac{1}{(4\pi)^{D\ell/2}} \int e^{-x^\tau A x} d^D x_1 \cdots d^D x_\ell = \det(A)^{-D/2}.$$

With the change of variable (2.12) and the conditions (2.13) and (2.14), one can rewrite the integral $U(\Gamma)$ of (2.2) in the form

$$U(\Gamma) = \int_{\mathbb{R}_+^n} e^{-V_\Gamma(t,p)} \left(\int e^{-x^\tau M_\Gamma(t) x} d^D x_i \cdots d^D x_\ell \right) dt_1\cdots dt_n, \tag{2.21}$$

with $\ell = b_1(\Gamma)$ is the number of loops in the graph. After performing the Gaussian integration and rewriting the expression in the external momenta as described above in the form (2.15) and (2.17), this becomes of the form

$$U(\Gamma) = (4\pi)^{-\ell D/2} \int_{\mathbb{R}_+^n} \frac{e^{-V_\Gamma(t,p)}}{\Psi_\Gamma(t)^{D/2}}\, dt_1\cdots dt_n, \tag{2.22}$$

with

$$\Psi_\Gamma(t) = \det M_\Gamma(t). \tag{2.23}$$

Then using the identity $1 = \int_0^\infty d\lambda\, \delta(\lambda - \sum_{i=1}^n t_i)$ and scaling $t_i \mapsto t_i\lambda$, one rewrites (2.22) in the form

$$U(\Gamma) = (4\pi)^{-\ell D/2} \int_0^\infty \left(\int_{[0,1]^n} \delta(1-\sum_i t_i) \frac{e^{-\lambda V_\Gamma(t,p)}}{\Psi_\Gamma(t)^{D/2}}\, dt_1\cdots dt_n \right) \lambda^{n-D\ell/2} \frac{d\lambda}{\lambda}. \tag{2.24}$$

Using again the special form

$$V_\Gamma^{-n+D\ell/2} = \frac{1}{\Gamma(n-D\ell/2)} \int_0^\infty e^{-\lambda V_\Gamma} \lambda^{n-D\ell/2-1}\, d\lambda \tag{2.25}$$

of the general identity (2.6), one then obtains the parametric form

$$U(\Gamma) = \frac{\Gamma(n-D\ell/2)}{(4\pi)^{\ell D/2}} \int_{[0,1]^n} \frac{\delta(1-\sum_i t_i)}{\Psi_\Gamma(t)^{D/2} V_\Gamma(t,p)^{n-D\ell/2}}\, dt_1\cdots dt_n. \tag{2.26}$$

The condition $n - D\ell/2 = \ell(1-D/2) + \#V(\Gamma) - 1 > 0$ ensures the convergence at $\lambda = 0$ of the integral (2.25). □

A graph is said to be *log divergent* if $n = D\ell/2$, in which case the Feynman integral reduces to the simpler form

$$\int_\Sigma \frac{\omega}{\det(M_\Gamma(t))^{D/2}}, \tag{2.27}$$

with $\omega = \delta(1-\sum_i t_i)dt_1 \cdots dt_n$ the volume form on the simplex Σ defined by the integration (2.26).

Remark 2.2. For the purpose of establishing relations between values of Feynman integrals and periods of motives, it is important to check that the multiplicative constant one is neglecting in passing from (2.2) to the parametric form (2.19) in fact belongs to $\mathbb{Q}(\pi)$, *cf.* [14]. In (2.26) one sees in fact that the multiplicative constant is of the form $\Gamma(n - D\ell/2)(4\pi)^{-\ell D/2}$. This either gives a divergent factor, at the poles of the Gamma function, in which case one considers the residue, or else, when convergent, it gives a multiplicative factor in $\mathbb{Q}(\pi)$.

The function $\Psi_\Gamma(t) = \det(M_\Gamma(t))$ has an equivalent expression in terms of the connectivity of the graph Γ as the polynomial (see [32], §6-2-3 and [38] §1.3-2)

$$\Psi_\Gamma(t) = \sum_S \prod_{e \in S} t_e, \tag{2.28}$$

where S ranges over all the sets $S \subset E_{int}(\Gamma)$ of $\ell = b_1(\Gamma)$ internal edges of Γ, such that the removal of all the edges in S leaves a connected graph. This can be equivalently formulated in terms of spanning trees of the graph Γ ([38] §1.3), *i.e.* $\Psi_\Gamma(t)$ is given by the Kirchhoff polynomial

$$\Psi_\Gamma(t) = \sum_T \prod_{e \notin T} t_e, \tag{2.29}$$

with the sum over spanning trees T of the graph. Each spanning tree, in fact, has $\#V - 1$ edges and is the complement of a cut-set S.

Lemma 2.3. *The graph polynomial Ψ_Γ is a homogeneous polynomial of degree*

$$\deg \Psi_\Gamma = b_1(\Gamma). \tag{2.30}$$

In the massless case *with $m = 0$, the function $V_\Gamma(t,p)$, for fixed p, is homogeneous of degree one and given by the ratio of a homogeneous polynomial $P_\Gamma(t,p)$ by $\Psi_\Gamma(t)$.*

Proof. We have $\deg \Psi_\Gamma = \#E(\Gamma) - \#E(T)$, where $\#E(T) = \#V(\Gamma) - 1$ is the nuber of edges in a (hence any) spanning tree, hence from the Euler characteristic formula $\#V(\Gamma) - \#E(\Gamma) = 1 - b_1(\Gamma)$ we get (2.30). We write the polynomial $V_\Gamma(t,p) = p^\tau R_\Gamma(t) p + \sum_i t_i m^2$. In the massless case, using the reformulation given in (6-87) and (6-88) of [32], p.297, we rewrite the function $V_\Gamma(t,p)$ in the form of the ratio

$$V_\Gamma(t,p) = \frac{P_\Gamma(t,p)}{\Psi_\Gamma(t)} \tag{2.31}$$

of a homogeneous polynomial P_Γ of degree $\ell + 1 = b_1(\Gamma) + 1$, divided by the polynomial Ψ_Γ, which is homogeneous of degree $b_1(\Gamma)$. In fact, we have ([32], §6-2-3)

$$P_\Gamma(p,t) = \sum_{C \subset \Gamma} s_C \prod_{e \in C} t_e, \tag{2.32}$$

where the sum is over the cut-sets $C \subset \Gamma$, *i.e.* the collections of $b_1(\Gamma) + 1$ edges that divide the graph Γ in exactly two connected components $\Gamma_1 \cup \Gamma_2$. The coefficient s_C is a function of the external momenta attached to the vertices in either one of the two components

$$s_C = \left(\sum_{v \in V(\Gamma_1)} P_v \right)^2 = \left(\sum_{v \in V(\Gamma_2)} P_v \right)^2, \tag{2.33}$$

where the P_v are defined as in (2.18), as the sum of the incoming external momenta (see [32], (6-87) and (6-88)). $\square$

In the following, we work under the following assumption on the graph Γ.

Definition 2.4. *A 1PI graph Γ satisfies the* generic condition *on the external momenta if, for p in a dense open set in the space of external momenta, the polynomials $P_\Gamma(t,p)$ and $\Psi_\Gamma(t)$ have no common factor.*

To understand better the nature of this condition, it is useful to reformulate the polynomial $P_\Gamma(t,p)$ of (2.32) in terms of spanning trees of the graph. One has, in the case where $m = 0$,

$$P_\Gamma(p,t) = \sum_T \sum_{e' \in T} s_{T,e'}\, t_{e'} \prod_{e \in T^c} t_e, \tag{2.34}$$

where $s_{T,e'} = s_C$ for the cut-set $C = T^c \cup \{e'\}$.

The parameterizing space of the external momenta is the hyperplane in the affine space $\mathbb{A}^{D \cdot \# E_{ext}(\Gamma)}$ obtained by imposing the conservation law

$$\sum_{e \in E_{ext}(\Gamma)} p_e = 0. \tag{2.35}$$

Thus, the simplest possible configuration of external momenta is the one where one puts all the external momenta to zero, except for a pair $p_{e_1} = p = -p_{e_2}$ associated to a choice of a pair of external edges $\{e_1, e_2\} \subset E_{ext}(\Gamma)$. Let v_i be the unique vertex attached to the external edge e_i of the chosen pair. We then have, in this case, $P_{v_1} = p = -P_{v_2}$. Upon writing the polynomial $P_\Gamma(t,p)$ in the form (2.34), we obtain in this case

$$P_\Gamma(p,t) = p^2 \sum_T \Big(\sum_{e' \in T_{v_1,v_2}} t_{e'} \Big) \prod_{e \notin T} t_e, \tag{2.36}$$

where $T_{v_1,v_2} \subset T$ is the unique path in T without backtrackings connecting the vertices v_1 and v_2. We use (2.33) to get $s_C = p^2$ for all the nonzero terms in this (2.36). These are all the terms that corresponds to cut sets C such that the vertices v_1 and v_2 belong to different components. These cut-sets consist of the complement of a spanning tree T and an edge of T_{v_1,v_2}.

In the following we will make use of the notation

$$L_T(t) = p^2 \sum_{e \in T_{v_1,v_2}} t_e \tag{2.37}$$

for the linear functions in (2.36).

If the polynomial $\Psi_\Gamma(t)$ of (2.29) divides (2.36), one has

$$P_\Gamma(p,t) = \Psi_\Gamma(t) \cdot L(t),$$

for a degree one polynomial $L(t)$, which gives

$$\sum_T (L_T(t) - L(t)) \prod_{e \notin T} t_e \equiv 0,$$

for all t. One then sees, for example, that the 1PI condition on the graph Γ is necessary in order to have the condition of Definition 2.4. In fact, for a graph that is not 1PI, one may be able to find vertices and momenta as above such that the degree one polynomials $L_T(t)$ are all equal to the same $L(t)$. Generally, the validity of the condition of Definition 2.4 can be checked algorithmically for a given graph.

One does not need to assume the condition of Definition 2.4. However, several of our formulae become more complicated if we allow the case where the polynomials Ψ_Γ and $P_\Gamma(t,p)$ have common factors. Thus, for our purposes we assume to work under the hypothesis that the "generic condition on the external momenta" holds.

Definition 2.5. *The affine graph hypersurface $\hat{X}_\Gamma$ is the zero locus of the Kirchhoff polynomial*

$$\hat{X}_\Gamma = \{t \in \mathbb{A}^n \,:\, \Psi_\Gamma(t) = 0\}, \tag{2.38}$$

with $n = \#E_{int}(\Gamma)$. The locus of zeros of the polynomial $P_\Gamma(t,p)$, for fixed external momenta p, also defines a hypersurface

$$\hat{Y}_\Gamma = \hat{Y}_\Gamma(p) := \{t \in \mathbb{A}^n \mid P_\Gamma(t,p) = 0\}. \tag{2.39}$$

Since both $\Psi_\Gamma(t)$ and $P_\Gamma(t,p)$ are homogeneous polynomials in t, we can consider corresponding projective hypersurfaces

$$X_\Gamma = \{t = (t_1 : \cdots : t_n) \in \mathbb{P}^{n-1} \,:\, \Psi_\Gamma(t) = 0\} \tag{2.40}$$

of degree $b_1(\Gamma)$ and

$$Y_\Gamma = Y_\Gamma(p) := \{t = (t_1 : \cdots : t_n) \in \mathbb{P}^{n-1} \mid P_\Gamma(t,p) = 0\}. \tag{2.41}$$

of degree $b_1(\Gamma) + 1$.

In the case of log divergent graphs, or of arbitrary graphs in the range with sufficiently large spacetime dimension D (*i.e.* for D satisfying $-n + D\ell/2 \geq 0$, with $n = \#E_{int}(\Gamma)$ and $\ell = b_1(\Gamma)$), the possible divergences of the Feynman integral $U(\Gamma)$ depend on the intersection of the domain of integration Σ with the graph hypersurface $\hat{X}_\Gamma$ in $\mathbb{P}^{n-1}$. Notice that the intersections $\Sigma \cap \hat{X}_\Gamma$ can only happen on the boundary $\partial\Sigma$, as in the interior of Σ the polynomial Ψ_Γ takes strictly positive real values. See [14] and [13] for a detailed analysis of this case and for its motivic interpretation. More generally, for non-log-divergent integrals of the form (2.19), outside of the range $-n + D\ell/2 \geq 0$, the singularities of the integral also involve the intersections of the hypersurfaces $\hat{Y}_\Gamma(t,p)$ with the domain of integration Σ. This case requires in general a more detailed analysis, as in this case some of the intersections may also appear away from the boundary of Σ, depending on the values of the external momenta p, see *e.g.* [11], §18.

2.2. **Dimensional Regularization.** One of the main problems that emerged in the historic development of perturbative quantum field theory is how to "cure" the divergences that occur systematically in the Feynman integrals (2.2), *i.e.* the problem of renormalization. Usually this is treated by choosing a *regularization* method, combined with a *renormalization* procedure. Regularization replaces a divergent integral (2.2) with a function of additional parameters that happens to have a pole or singularity at the special value of the parameter that corresponds to the original integral, but which is otherwise well defined and finite at nearby values of the parameter. Renormalization, on the other hand, gives a method for extracting finite values from the regularized expressions in a way that is consistent with the combinatorics of nested subdivergences, *i.e.* subgraphs of graphs with divergent Feynman integrals, which themselves contribute divergences.

The Connes–Kreimer theory [21] uses the regularization method known as *dimensional regularization and minimal subtraction*, combined with the renormalization procedure of Bogoliubov–Parasiuk–Hepp–Zimmermann (BPHZ). It was later shown (see *e.g.* [28]) that the main results of Connes–Kreimer may be applied to other regularization procedures, as long as the "subtraction of infinities" can be formulated in terms of a Rota–Baxter operator. The projection of a Laurent series onto its polar part is an example of such an operator, which corresponds to the "minimal subtraction" case. Using this more general formulation, it possible to extend the Connes–Kreimer theory to other regularization methods, which makes it possible, for instance, to extend it to the case of curved backgrounds as in [1]. We concentrate here on the Dimensional Regularization and Minimal Subtraction procedure. In fact, our purpose is to compare the approach to motives and

renormalization of [22] with the one of [14], and we prefer to remain close to the formulation given in [22] using DimReg.

Dimensional Regularization consists of formally extending the usual Gaussian integration (2.20) from the case of integer dimension $D \in \mathbb{N}$ to the case of a "complexified dimension" $z \in \mathbb{C}$, in a small neighborhood of $z = 0$, by setting

$$\int e^{-\frac{1}{2}x^\tau A x} d^{D+z}x_1 \cdots d^{D+z}x_\ell := \frac{(2\pi)^{(D+z)\ell/2}}{\det(A)^{(D+z)/2}}, \tag{2.42}$$

This results is the analytic continuation of the parametric Feynman integral formulae (2.22), (2.24), (2.26) to complex values of the dimension D.

Lemma 2.6. *Upon replacing the integer dimension D by a complexified dimension $D \mapsto D + z$, with $z \in \Delta^*$ a small punctured disk around $z = 0$, the integral* (2.21) *becomes of the form*

$$U(\Gamma)(z) = \frac{\Gamma(n - \frac{(D+z)\ell}{2})}{(4\pi)^{\frac{\ell(D+z)}{2}}} \int_{[0,1]^n} \frac{\delta(1-\sum_i t_i)}{\Psi_\Gamma(t)^{(D+z)/2} V_\Gamma(t,p)^{n-(D+z)\ell/2}} \, dt_1 \cdots dt_n. \tag{2.43}$$

Proof. One uses the same argument of Lemma 2.1, but using (2.42) instead of (2.20) in (2.21). This gives

$$U(\Gamma) = (4\pi)^{-\ell(D+z)/2} \int_{\mathbb{R}_+^n} \frac{e^{-V_\Gamma(t,p)}}{\Psi_\Gamma(t)^{(D+z)/2}} \, dt_1 \cdots dt_n, \tag{2.44}$$

We then use the same argument as in Lemma 2.1 to write this in the form

$$U(\Gamma) = (4\pi)^{-\ell(D+z)/2} \int_0^\infty \left(\int_{[0,1]^n} \delta(1 - \sum_i t_i) \frac{e^{-\lambda V_\Gamma(t,p)}}{\Psi_\Gamma(t)^{(D+z)/2}} \, dt_1 \cdots dt_n \right) \lambda^{n-(D+z)\ell/2} \frac{d\lambda}{\lambda} \tag{2.45}$$

and we use

$$V_\Gamma^{-n+(D+z)\ell/2} = \frac{1}{\Gamma(n-(D+z)\ell/2)} \int_0^\infty e^{-\lambda V_\Gamma} \lambda^{n-(D+z)\ell/2-1} \, d\lambda \tag{2.46}$$

to obtain (2.43). One recovers the parametric form (2.19) from (2.42). $\square$

2.3. **Mass scale dependence.** It is well known that, when one regularizes the integrals $U(\Gamma)$ using dimensional regularization, as recalled above, one introduces an explicit dependence on the mass scale, which plays a very important role in the renormalization process and is the source of the nontrivial action of the renormalization group (see [20], [21], [22], [23]).

The source of the mass scale dependence is the fact that, in order to maintain the physical units, the integral (2.42) should in fact be written in the form

$$\int e^{-\frac{1}{2}x^\tau A x} \mu^{-z} d^{D+z}x_1 \cdots \mu^{-z} d^{D+z}x_\ell := \mu^{-z\ell} \frac{(2\pi)^{(D+z)\ell/2}}{\det(A)^{(D+z)/2}}, \tag{2.47}$$

where μ has the physical units of a mass (energy), so that the $\mu^{-z} d^{D+z}x_i$ still have the same physical units as the original $d^D x_i$ (see [20]).

Lemma 2.7. *The dimensional regularization $U(\Gamma)(z)$ of* (2.43) *depends on the mass scale μ in the form*

$$U_\mu(\Gamma)(z) = \mu^{-z\ell} \frac{\Gamma(n - \frac{(D+z)\ell}{2})}{(4\pi)^{\frac{\ell(D+z)}{2}}} \int_{[0,1]^n} \frac{\delta(1-\sum_i t_i) dt_1 \cdots dt_n}{\Psi_\Gamma(t)^{\frac{(D+z)}{2}} V_\Gamma(t,p)^{n-\frac{(D+z)\ell}{2}}} \, . \tag{2.48}$$

Proof. In the derivation of the parametric form of the Feynman integral with dimensional regularization, we see that we have in (2.44) a mass scale dependence

$$U_\mu(\Gamma)(z) = (4\pi)^{-\ell(D+z)/2} \mu^{-z\ell} \int_{\mathbb{R}_+^n} \frac{e^{-V_\Gamma(t,p)}}{\Psi_\Gamma(t)^{(D+z)/2}}\, dt_1 \cdots dt_n. \tag{2.49}$$

The rest of the argument of Lemma 2.6 is unchanged. In particular, no further μ dependence is introduced by the term in $V_\Gamma(t,p)$, so that we obtain (2.48). □

2.4. **Integrals on projective spaces.** As remarked above, due to the homogeneity of the polynomials Ψ_Γ and P_Γ, it is natural to regard the graph hypersurfaces as projective hypersurfaces X_Γ and Y_Γ in $\mathbb{P}^{n-1}$, with n the number of internal lines of the graph Γ. Thus, we want to think of the parametric Feynman integrals as being computed in projective space.

In order to reformulate in projective space $\mathbb{P}^{n-1}$ integrals originally defined in affine space $\mathbb{A}^n$, one needs to work with the projective analog (*cf.* [29], §II) of the volume form

$$\omega_n = dt_1 \wedge \cdots \wedge dt_n.$$

This is given by the form

$$\Omega = \sum_{i=1}^n (-1)^{i+1} t_i\, dt_1 \wedge \cdots \wedge \widehat{dt_i} \wedge \cdots \wedge dt_n. \tag{2.50}$$

The relation between the volume form $dt_1 \wedge \cdots \wedge dt_n$ and the homogeneous form Ω of degree n of (2.50) is given by (*cf.* [26], p.180)

$$\Omega = \Delta(\omega_n), \tag{2.51}$$

where $\Delta : \Omega^k \to \Omega^{k-1}$ is the operator of contraction with the Euler vector field

$$E = \sum_i t_i \frac{\partial}{\partial t_i}, \tag{2.52}$$

$$\Delta(\omega)(v_1, \cdots, v_{k-1}) = \omega(E, v_1, \cdots, v_{k-1}). \tag{2.53}$$

In the parametric Feynman integrals, we consider as region of definition of the integrand (in the log divergent case, or in the case of integrals in the range $-n + D\ell/2 \geq 0$) the hypersurface complement

$$\mathcal{D}(\Psi_\Gamma) = \{t \in \mathbb{A}^n \mid \Psi_\Gamma(t) \neq 0\} = \mathbb{A}^n \smallsetminus \hat{X}_\Gamma, \tag{2.54}$$

while in the formulation (2.26) outside of the range $-n + D\ell/2 \geq 0$, we also need to avoid the second hypersurface $\hat{Y}_\Gamma$ defined by the vanishing of P_Γ (for assigned external momenta), as in (2.39). In this case the domain of definition of the integrand is

$$\begin{aligned} \mathcal{D}(\Psi_\Gamma, P_\Gamma) = &\quad \{t \in \mathbb{A}^n \mid \Psi_\Gamma(t) \neq 0 \text{ and } P_\Gamma(t,p) \neq 0\} \\ = &\quad \mathcal{D}(\Psi_\Gamma) \cap \mathcal{D}(P_\Gamma) = \mathbb{A}^n \smallsetminus (\hat{X}_\Gamma \cup \hat{Y}_\Gamma). \end{aligned} \tag{2.55}$$

Let $\mathcal{U}(\Psi_\Gamma)$ and $\mathcal{U}(\Psi_\Gamma, P_\Gamma)$ denote the corresponding hypersurface complements in projective space, namely

$$\begin{aligned} \mathcal{U}(\Psi_\Gamma) = &\quad \{t \in \mathbb{P}^{n-1} \mid \Psi_\Gamma(t) \neq 0\} = \mathbb{P}^{n-1} \smallsetminus X_\Gamma \\ \mathcal{U}(\Psi_\Gamma, P_\Gamma) = &\quad \{t \in \mathbb{P}^{n-1} \mid \Psi_\Gamma(t) \neq 0 \text{ and } P_\Gamma(t,p) \neq 0\} \\ = &\quad \mathcal{U}(\Psi_\Gamma) \cap \mathcal{U}(P_\Gamma) = \mathbb{P}^{n-1} \smallsetminus (X_\Gamma \cup Y_\Gamma). \end{aligned} \tag{2.56}$$

As we see in more detail in (2.70) and Proposition 2.9 below, in both the affine and the projective case, we can describe $\mathcal{D}(\Psi_\Gamma, P_\Gamma)$ and $\mathcal{U}(\Psi_\Gamma, P_\Gamma)$ as hypersurface complements, by identifying $X_\Gamma \cup Y_\Gamma$ with the hypersurface defined by the vanishing of a homogeneous polynomial given by a product $\Psi_\Gamma^{n_1} \cdot P_\Gamma^{n_2}$, a homogeneous polynomial of degree $n_1 b_1(\Gamma) + n_2(b_1(\Gamma)+1)$, where the component hypersurfaces X_Γ and Y_Γ are counted with multiplicities n_1 and n_2. These multiplicities depend on the number of edges and loops of the graph and on the spacetime dimension, and are defined more precisely in (2.70) below. Thus, in the following, wherever needed, we write $\mathcal{D}(\Psi_\Gamma, P_\Gamma) = \mathcal{D}(f)$ and $\mathcal{U}(\Psi_\Gamma, P_\Gamma) = \mathcal{U}(f)$, with $f = \Psi_\Gamma^{n_1} \cdot P_\Gamma^{n_2}$, as in the various cases of (2.70) below.

We introduce here some notation that will be useful in the following (*cf.* [26], p.177). Let $\mathcal{R} = \mathbb{C}[t_1, \dots, t_n]$ be the ring of polynomials of $\mathbb{A}^n$. Let $\mathcal{R}_m$ denotes the subset of homogeneous polynomials of degree m. Similarly, let Ω^k denote the $\mathcal{R}$-module of k-forms on $\mathbb{A}^n$ and let Ω^k_m denote the subset of k-forms that are homogeneous of degree m.

We recall the following general fact (*cf.* [26], p.178) about hypersurface complements. Let $\pi : \mathbb{A}^n \smallsetminus \{0\} \to \mathbb{P}^{n-1}$ be the standard projection $t = (t_1, \dots, t_n) \mapsto t = (t_1 : \cdots : t_n)$. Suppose given a homogeneous polynomial function f on $\mathbb{A}^n$ of degree $\deg(f)$. Let $\mathcal{D}(f) \subset \mathbb{A}^n$ and $\mathcal{U}(f) \subset \mathbb{P}^{n-1}$ be the hypersurface complements, *i.e.* the complements, in $\mathbb{A}^n$ and $\mathbb{P}^{n-1}$ respectively, of the locus of zeros $X_f = \{t \,|\, f(t) = 0\}$. With the notation introduced here above, we can always write a form $\omega \in \Omega^k(\mathcal{D}(f))$ as

$$\omega = \frac{\eta}{f^m}, \quad \text{with} \quad \eta \in \Omega^k_{m\deg(f)}. \tag{2.57}$$

We then have the following characterization of the pullback along $\pi : \mathcal{D}(f) \to \mathcal{U}(f)$ of forms on $\mathcal{U}(f)$ (see [26], p.180 and [27]). Given $\omega \in \Omega^k(\mathcal{U}(f))$, the pullback $\pi^*(\omega) \in \Omega^k(\mathcal{D}(f))$ is characterized by the properties of being invariant under the $\mathbb{G}_m$ action on $\mathbb{A}^n \smallsetminus \{0\}$ and of satisfying $\Delta(\pi^*(\omega)) = 0$, where Δ is the contraction (2.53) with the Euler vector field E of (2.52). Thus, since the sequence

$$0 \to \Omega^n \xrightarrow{\Delta} \Omega^{n-1} \xrightarrow{\Delta} \cdots \Omega^1 \xrightarrow{\Delta} \Omega^0 \to 0$$

is exact at all but the last term, one can write

$$\pi^*(\omega) = \frac{\Delta(\eta)}{f^m}, \quad \text{with} \quad \eta \in \Omega^k_{m\deg(f)}. \tag{2.58}$$

Thus, in particular, any $(n-1)$-form on $\mathcal{U}(f) \subset \mathbb{P}^{n-1}$ can be written as

$$\frac{h\Omega}{f^m}, \quad \text{with} \quad h \in \mathcal{R}_{m\deg(f)-n} \tag{2.59}$$

and with $\Omega = \Delta(dt_1 \wedge \cdots \wedge dt_n)$ the $(n-1)$-form (2.50), homogeneous of degree n.

Proposition 2.8. *Let $\omega \in \Omega^k_{m\deg(f)}$ be a closed k-form, which is homogeneous of degree $m\deg(f)$, and consider the form ω/f^m on $\mathbb{A}^n$. Let $\Sigma \subset \mathbb{A}^n \smallsetminus \{0\}$ be a k-dimensional domain with boundary $\partial\Sigma \neq \emptyset$. Then the integration of ω/f^m on Σ satisfies*

$$m\deg(f) \int_\Sigma \frac{\omega}{f^m} = \int_{\partial\Sigma} \frac{\Delta(\omega)}{f^m} + \int_\Sigma df \wedge \frac{\Delta(\omega)}{f^{m+1}}. \tag{2.60}$$

Proof. Recall that we have ([26], [27])

$$d\left(\frac{\Delta(\omega)}{f^m}\right) = -\frac{\Delta(d_f\omega)}{f^{m+1}}, \tag{2.61}$$

where, for a form ω that is homogeneous of degree $m\deg(f)$,

$$d_f\omega = f\,d\omega - m\,df \wedge \omega. \tag{2.62}$$

Thus, we have

$$d\left(\frac{\Delta(\omega)}{f^m}\right) = -\frac{\Delta(d\omega)}{f^m} + m\frac{\Delta(df \wedge \omega)}{f^{m+1}}. \tag{2.63}$$

Since the form ω is closed, $d\omega = 0$, and we have

$$\Delta(df \wedge \omega) = \deg(f)\, f\, \omega - df \wedge \Delta(\omega), \tag{2.64}$$

we obtain from the above

$$d\left(\frac{\Delta(\omega)}{f^m}\right) = m \deg(f) \frac{\omega}{f^m} - \frac{df \wedge \Delta(\omega)}{f^{m+1}}. \tag{2.65}$$

By Stokes' theorem we have

$$\int_{\partial\Sigma} \frac{\Delta(\omega)}{f^m} = \int_\Sigma d\left(\frac{\Delta(\omega)}{f^m}\right).$$

Using (2.65) this gives

$$\int_{\partial\Sigma} \frac{\Delta(\omega)}{f^m} = m \deg(f) \int_\Sigma \frac{\omega}{f^m} - \int_\Sigma \frac{df \wedge \Delta(\omega)}{f^{m+1}}. \tag{2.66}$$

$\square$

We can use this result to reformulate the parametric Feynman integrals in terms of integrals of forms that are pullbacks to $\mathbb{A}^n \smallsetminus \{0\}$ of forms on a hypersurface complement in $\mathbb{P}^{n-1}$. For simplicity, we remove here the divergent Γ-factor from the parametric Feynman integral and we concentrate on the residue given by the integration on the simplex Σ as in (2.67) below.

Proposition 2.9. *Under the generic condition on the external momenta, the parametric Feynman integral*

$$\mathbb{U}(\Gamma) = \int_\Sigma \frac{\omega_n}{\Psi_\Gamma^{D/2} V_\Gamma^{n-D\ell/2}} \tag{2.67}$$

can be computed as

$$\mathbb{U}(\Gamma) = \frac{1}{C(n,D,\ell)} \left(\int_{\partial\Sigma} \pi^*(\eta) + \int_\Sigma df \wedge \frac{\pi^*(\eta)}{f} \right), \tag{2.68}$$

where $\pi : \mathbb{A}^n \smallsetminus \{0\} \to \mathbb{P}^{n-1}$ is the projection and η is the form on the hypersurface complement $\mathcal{U}(f)$ in $\mathbb{P}^{n-1}$ with

$$\pi^*(\eta) = \frac{\Delta(\omega)}{f^m}, \tag{2.69}$$

on $\mathbb{A}^n$, where

$$f = \begin{cases} P_\Gamma & n - \frac{D(\ell+1)}{2} \geq 0 \\ P_\Gamma^{\frac{2n-D\ell}{2m}} \Psi_\Gamma^{\frac{D}{2m}} & n - \frac{D(\ell+1)}{2} < 0 < n - \frac{D\ell}{2} \\ & m = \gcd\{n - D\ell/2, D/2\} \\ \Psi_\Gamma & n - \frac{D\ell}{2} \leq 0, \end{cases} \tag{2.70}$$

with

$$m = \begin{cases} n - D\ell/2 & n - \frac{D(\ell+1)}{2} \geq 0 \\ \gcd\{n - D\ell/2, D/2\} & n - \frac{D(\ell+1)}{2} < 0 < n - \frac{D\ell}{2} \\ -n + D(\ell+1)/2 & n - \frac{D\ell}{2} \leq 0, \end{cases} \tag{2.71}$$

and with

$$\omega = \begin{cases} \Psi_\Gamma^{n-D(\ell+1)/2}\omega_n & n - \frac{D(\ell+1)}{2} \geq 0 \\ \Psi_\Gamma^{n-D\ell/2}\omega_n & n - \frac{D(\ell+1)}{2} < 0 < n - \frac{D\ell}{2} \\ P_\Gamma^{-n+D\ell/2}\omega_n & n - \frac{D\ell}{2} \leq 0, \end{cases} \tag{2.72}$$

where $\omega_n = dt_1 \wedge \cdots \wedge dt_n$ *on* $\mathbb{A}^n$*, with* $\Omega = \Delta(\omega_n)$ *as in* (2.50)*. The coefficient* $C(n, D, \ell)$ *in* (2.68) *is given by*

$$C(n, D, \ell) = \begin{cases} (n - D\ell/2)(\ell + 1) & n - \frac{D(\ell+1)}{2} \geq 0 \\ (n - D\ell/2)\ell + n & n - \frac{D(\ell+1)}{2} < 0 < n - \frac{D\ell}{2} \\ -(n - D(\ell + 1)/2)\ell & n - \frac{D\ell}{2} \leq 0. \end{cases} \tag{2.73}$$

Proof. Consider on $\mathbb{A}^n$ the form given by $\Delta(\omega)/f^m$, with f, m, and ω, respectively as in (2.70), (2.71) and (2.72). We assume the condition of Definition 2.4, *i.e.* for a generic choice of the external momenta the polynomials P_Γ and Ψ_Γ have no common factor. First notice that, since the polynomial Ψ_Γ is homogeneous of degree ℓ and P_Γ is homogeneous of degree $\ell + 1$, the form $\Delta(\omega)/f^m$ is $\mathbb{G}_m$ invariant on $\mathbb{A}^n \smallsetminus \{0\}$. Moreover, since it is of the form $\alpha = \Delta(\omega)/f^m$, it also satisfies $\Delta(\alpha) = 0$, hence it is the pullback of a form η on $\mathcal{U}(f) \subset \mathbb{P}^{n-1}$. Also notice that the domain of integration $\Sigma \subset \mathbb{A}^n$ given by the simplex $\Sigma = \{\sum_i t_i = 1, t_i \geq 0\}$, is contained in a fundamental domain of the action of the multiplicative group $\mathbb{C}^*$ on $\mathbb{C}^n \smallsetminus \{0\}$.

Applying the result of Proposition 2.8 above, we obtain

$$\begin{aligned} \int_\Sigma \frac{dt_1 \wedge \cdots \wedge dt_n}{\Psi_\Gamma^{D/2} V_\Gamma^{n-D\ell/2}} &= \int_\Sigma \frac{\omega}{f^m} \\ &= \frac{1}{m \deg(f)} \left(\int_{\partial\Sigma} \frac{\Delta(\omega)}{f^m} + \int_\Sigma df \wedge \frac{\Delta(\omega)}{f^{m+1}} \right) \\ &= C(n, D, \ell)^{-1} \left(\int_{\partial\Sigma} \frac{\Delta(\omega_n)}{\Psi_\Gamma^{D/2} V_\Gamma^{n-D\ell/2}} + \int_\Sigma df \wedge \frac{\Delta(\omega_n)}{\Psi_\Gamma^a P_\Gamma^b} \right), \end{aligned}$$

where f is as in (2.70) and

$$a = \begin{cases} D(\ell+1)/2 - n & n - \frac{D(\ell+1)}{2} \geq 0 \\ \frac{D}{2}(1 + \frac{1}{m}) & n - \frac{D(\ell+1)}{2} < 0 < n - \frac{D\ell}{2} \\ -n + \frac{D(\ell+1)}{2} + 1 & n - \frac{D\ell}{2} \leq 0, \end{cases} \tag{2.74}$$

$$b = \begin{cases} n - \frac{D\ell}{2} + 1 & n - \frac{D(\ell+1)}{2} \geq 0 \\ (n - \frac{D\ell}{2})(1 + \frac{1}{m}) & n - \frac{D(\ell+1)}{2} < 0 < n - \frac{D\ell}{2} \\ n - \frac{D\ell}{2} & n - \frac{D\ell}{2} \leq 0. \end{cases} \tag{2.75}$$

In fact, the cases of $n - \frac{D(\ell+1)}{2} \geq 0$ and $n - \frac{D\ell}{2} \leq 0$ are clear, while in the range with $n - \frac{D(\ell+1)}{2} < 0$ and $n - \frac{D\ell}{2} > 0$ we have

$$f^{m+1} = P_\Gamma^{(n-D\ell/2)(1+\frac{1}{m})} \Psi_\Gamma^{\frac{D}{2}(1+\frac{1}{m})}.$$

The coefficient $C(n, D, \ell)$ is given by $C(n, D, \ell) = m \deg(f)$, with m and f as in (2.71) and (2.70). Thus, it is given by (2.73), where in the second case we use

$$m((\ell + 1)(n - D\ell/2)/m + D\ell/2m) = (n - D\ell/2)\ell + n,$$

for $m = \gcd\{n - D\ell/2, D/2\}$. $\square$

3. Singularities, slicing, and Milnor fiber

3.1. **Non-isolated singularities.** One problem in trying to use in our setting the techniques developed in singularity theory (*cf.* [5]) to study mixed Hodge structures in terms of oscillatory integrals is that the graph hypersurfaces $X_\Gamma \subset \mathbb{P}^{n-1}$ defined by the vanishing of the polynomial $\Psi_\Gamma(t) = \det(M_\Gamma(t))$ usually have non-isolated singularities. This can be seen easily by the following observation.

Lemma 3.1. *Let Γ be a graph with $\deg \Psi_\Gamma > 2$. The singular locus of X_Γ is given by the intersection of cones over the hypersurfaces X_{Γ_e}, for $e \in E(\Gamma)$, where Γ_e is the graph obtained by removing the edge e of Γ. The cones $C(X_{\Gamma_e})$ do not intersect transversely.*

Proof. First observe that, since X_Γ is defined by a homogeneous equation $\Psi_\Gamma(t) = 0$, with Ψ_Γ a polynomial of degree m, the Euler formula $m\Psi_\Gamma(t) = \sum_e t_e \frac{\partial}{\partial t_e}\Psi_\Gamma(t)$ implies that $\cap_e Z(\partial_e \Psi_\Gamma) \subset X_\Gamma$, where $Z(\partial_e\Psi_\Gamma)$ is the zero locus of the t_e-derivative. Thus, the singular locus of X_Γ is just given by the equations $\partial_e \Psi_\Gamma = 0$. The variables t_e appear in the polynomial $\Psi_\Gamma(t)$ only with degree zero or one, hence the polynomial $\partial_e\Psi_\Gamma$ consists of only those monomials of Ψ_Γ that contain the variable t_e, where one sets $t_e = 1$. The resulting polynomial is therefore of the form Ψ_{Γ_e}, where Γ_e is the graph obtained from Γ by removing the edge e. In fact, one can see in terms of spanning trees that, if T is a spanning tree containing the edge e then $T \smallsetminus e$ is no longer a spanning tree of Γ_e, so the corresponding terms disappear in passing from Ψ_Γ to Ψ_{Γ_e}, while if T is a spanning tree of Γ which does not contain e, then T is still a spanning tree of Γ_e and the corresponding monomial m_T of Ψ_{Γ_e} is the same as the monomial m_T in Ψ_Γ without the variable t_e. Thus, the zero locus $Z(\Psi_{\Gamma_e}) \subset \mathbb{P}^{n-1}$ is a cone $C(X_{\Gamma_e})$ over the graph hypersurface $X_{\Gamma_e} \subset \mathbb{P}^{n-2}$ with vertex at the coordinate point $v_e = (0, \ldots, 0, 1, 0, \ldots 0)$ with $t_e = 1$. To see that these cones do not intersect transversely, notice that, in the case where $\deg \Psi_\Gamma > 2$, given any two $C(X_{\Gamma_e})$ and $C(X_{\Gamma_{e'}})$ the vertex of one cone is contained in the graph hypersurface spanning the other cone. $\square$

The work of Bergbauer–Rej [10] gives a more detailed analysis of the singular locus of the graph hypersurfaces, using a formula for the Kirchhoff polynomials under insertion of subgraphs at vertices.

3.2. **Projective Radon transform.** Among various techniques introduced for the study of non-isolated singularities, a common procedure consists of cutting the ambient space with linear spaces of dimension complementary to that of the singular locus of the hypersurface (*cf. e.g.* [42]). In this case, the restriction of the function defining the hypersurface to these linear spaces defines hypersurfaces with isolated singularities, to which the usual invariants and constructions for isolated singularities can be applied.

One finds that, in typical cases, the graph hypersurfaces have singular locus of codimension one, which means that the slicing is given by planes $\mathbb{P}^2$ intersecting the hypersurface into a curve with isolated singular points. When the singular locus is of codimension two in the hypersurface, the slicing is given by 3-dimensional spaces cutting the hypersurface into a family of surfaces in $\mathbb{P}^3$ with isolated singularities.

In our setting, we are interested in computing integrals of the form (2.67). From this point of view, the procedure of restricting the function defining the hypersurface to linear spaces of a fixed dimension correspond to an integral transform analogous to a Radon transform in projective space (*cf.* [29]).

We recall the basic setting for integral transforms on projective spaces (*cf.* §II of [29]). On any k-dimensional subspace $\mathbb{A}^k \subset \mathbb{A}^n$ there is a unique (up to a multiplicative constant)

$(k-1)$-form that is invariant under the action of SL_k. It is given as in (2.50) by the expression

$$\Omega_k = \sum_{i=1}^{k} (-1)^{i+1} t_i \, dt_1 \wedge \cdots \wedge \widehat{dt_i} \wedge \cdots \wedge dt_k. \tag{3.1}$$

The form (3.1) is homogeneous of degree k. Suppose given a function f on $\mathbb{A}^n$ which satisfies the homogeneity condition

$$f(\lambda t) = \lambda^{-k} f(t), \quad \forall t \in \mathbb{A}^n,\ \lambda \in \mathbb{G}_m. \tag{3.2}$$

Then the integrand $f\Omega_k$ is well defined on the corresponding projective space $\mathbb{P}^{k-1} \subset \mathbb{P}^{n-1}$ and one defines the integral as integrating on a fundamental domain in $\mathbb{A}^k \smallsetminus \{0\}$, *i.e.* on a surface that intersect each line from the origin once.

Suppose given dual vectors $\xi_i \in (\mathbb{A}^n)'$, for $i = 1, \ldots, n-k$. These define a k-dimensional linear subspace $\Pi = \Pi_\xi \subset \mathbb{A}^n$ by the vanishing

$$\Pi_\xi = \{t \in \mathbb{A}^n \mid \langle \xi_i, t\rangle = 0,\ i = 1, \ldots, n-k\}. \tag{3.3}$$

Given a choice of a subspace Π_ξ, there exists a $(k-1)$-form Ω_ξ on $\mathbb{A}^n$ satisfying

$$\langle \xi_1, dt\rangle \wedge \cdots \wedge \langle \xi_{n-k}, dt\rangle \wedge \Omega_\xi = \Omega_n, \tag{3.4}$$

with Ω_n the $(n-1)$-form of (2.50), *cf.* (3.1). The form Ω_ξ is not uniquely defined on $\mathbb{A}^n$, but its restriction to Π_ξ is uniquely defined by (3.4). Then, given a function f on $\mathbb{A}^n$ with the homogeneity condition (3.2), one can consider the integrand $f\Omega_\xi$ and define its integral on the projective space $\pi(\Pi_\xi) \subset \mathbb{P}^{n-1}$ as above. This defines the integral transform, that is, the $(k-1)$-dimensional projective Radon transform (§II of [29]) as

$$\mathcal{F}_k(f)(\xi) = \int_{\pi(\Pi_\xi)} f(t)\, \Omega_\xi(t) = \int_{\mathbb{P}^{n-1}} f(t) \prod_{i=1}^{n-k} \delta(\langle \xi_i, t\rangle)\, \Omega_\xi(t). \tag{3.5}$$

For our purposes, it is convenient to consider also the following variant of the Radon transform (3.5).

Definition 3.2. *Let $\Sigma \subset \mathbb{A}^n$ be a compact region that is contained in a fundamental domain of the action of $\mathbb{G}_m$ on $\mathbb{A}^n \smallsetminus \{0\}$. The partial $(k-1)$-dimensional projective Radon transform is given by the expression*

$$\mathcal{F}_{\Sigma,k}(f)(\xi) = \int_{\Sigma \cap \pi(\Pi_\xi)} f(t)\, \Omega_\xi(t) = \int_{\Sigma \cap \pi(\Pi_\xi)} f(t) \prod_{i=1}^{n-k} \delta(\langle \xi_i, t\rangle)\, \Omega_\xi(t), \tag{3.6}$$

where one identifies Σ with its image $\pi(\Sigma) \subset \mathbb{P}^{n-1}$.

Let us now return to the parametric Feynman integrals we are considering.

Proposition 3.3. *The Feynman integral* (2.22) *can be reformulated as*

$$U(\Gamma) = \frac{\Gamma(k - \frac{D\ell}{2})}{(4\pi)^{\ell D/2}} \int \mathcal{F}_{\Sigma,k}(f_\Gamma)(\xi)\, \langle \xi, dt\rangle, \tag{3.7}$$

where ξ is an $(n-k)$-frame in $\mathbb{A}^n$ and $\mathcal{F}_{\Sigma,k}(f)$ is the Radon transform, with Σ the simplex $\sum_i t_i = 1$, $t_i \geq 0$, and with

$$f_\Gamma(t) = \frac{V_\Gamma(t,p)^{-k+D\ell/2}}{\Psi_\Gamma(t)^{D/2}}. \tag{3.8}$$

Proof. Consider first the form (2.22) of the Feynman integral, which we write equivalently as

$$U(\Gamma) = (4\pi)^{-\ell D/2} \int_{\mathbb{A}^n} \chi_+(t) \frac{e^{-V_\Gamma(t,p)}}{\Psi_\Gamma(t)^{D/2}} \, dt_1 \cdots dt_n, \tag{3.9}$$

where $\chi_+(t)$ is the characteristic function of the domain $\mathbb{R}^n_+$.

Given a choice of an $(n-k)$-frame ξ, we can then write the Feynman integrals in the form

$$U(\Gamma) = (4\pi)^{-\ell D/2} \int \left(\int_{\Pi_\xi} \chi_+(t) \frac{e^{-V_\Gamma(t,p)}}{\Psi_\Gamma(t)^{D/2}} \omega_\xi \right) \langle \xi, dt \rangle, \tag{3.10}$$

where $\langle \xi, dt \rangle$ is a shorthand notation for

$$\langle \xi, dt \rangle = \langle \xi_1, dt \rangle \wedge \cdots \wedge \langle \xi_{n-k}, dt \rangle$$

and ω_ξ satisfies

$$\langle \xi, dt \rangle \wedge \omega_\xi = \omega_n = dt_1 \wedge \cdots \wedge dt_n. \tag{3.11}$$

We then apply the same procedure as in (2.24) and (2.25) to the integral on Π_ξ and write it in the form

$$\int_{\Pi_\xi} \chi_+(t) \frac{e^{-V_\Gamma(t,p)}}{\Psi_\Gamma(t)^{D/2}} \omega_\xi(t) = \Gamma(k - \frac{D\ell}{2}) \int_{\Pi_\xi} \delta(1 - \sum_i t_i) \, \frac{\omega_\xi(t)}{\Psi_\Gamma(t)^{D/2} V_\Gamma(t,p)^{k - D\ell/2}}. \tag{3.12}$$

The function $f_\Gamma(t)$ of (3.8) satisfies the scaling property (3.2) and the integrand

$$\frac{\omega_\xi(t)}{\Psi_\Gamma(t)^{D/2} V_\Gamma(t,p)^{k-D\ell/2}}$$

is therefore $\mathbb{G}_m$-invariant, since the form ω_ξ is homogeneous of degree k. Moreover, the domain Σ of integration is contained in a fundamental domain for the action of $\mathbb{G}_m$. Thus, we can reformulate the integral (3.12) in projective space, in terms of Radon transform as

$$\Gamma(k - \frac{D\ell}{2}) \, (4\pi)^{-\ell D/2} \int \mathcal{F}_{\Sigma,k}(f_\Gamma)(\xi) \, \langle \xi, dt \rangle, \tag{3.13}$$

where $\mathcal{F}_{\Sigma,k}(f_\Gamma)$ is the Radon transform over the simplex Σ, as in Definition 3.2. $\square$

In the following, we will then consider integrals of the form

$$\mathbb{U}(\Gamma)_\xi = \mathcal{F}_{\Sigma,k}(f_\Gamma)(\xi) = \int_{\Pi_\xi} \delta(1 - \sum_i t_i) \, \frac{\omega_\xi(t)}{\Psi_\Gamma(t)^{D/2} V_\Gamma(t,p)^{k-D\ell/2}} \tag{3.14}$$

$$= \int_{\Sigma_\xi} \frac{\omega_\xi(t)}{\Psi_\Gamma(t)^{D/2} V_\Gamma(t,p)^{k-D\ell/2}}$$

as well as their dimensional regularizations

$$\mathbb{U}(\Gamma)_\xi(z) = \int_{\Sigma_\xi} \frac{\omega_\xi(t)}{\Psi_\Gamma(t)^{(D+z)/2} V_\Gamma(t,p)^{k-(D+z)\ell/2}}, \tag{3.15}$$

where Π_ξ is a generic linear subspace of dimension equal to the codimension of the singular locus of the hypersurface $X_\Gamma \cup Y_\Gamma$.

3.3. **The polar filtration.** As we recalled already in §2.4 above (*cf.* [26]) algebraic differential forms $\omega \in \Omega^k(\mathcal{D}(f))$ on a hypersurface complement can always be written in the form $\omega = \eta/f^m$ as in (2.57), for some $m \in \mathbb{N}$ and some $\eta \in \Omega^k_{m\deg(f)}$. The minimal m such that ω can be written in the form $\omega = \eta/f^m$ is called the order of pole of ω along the hypersurface X and is denoted by $\mathrm{ord}_X(\omega)$. The order of pole induces a filtration, called the *polar filtration*, on the de Rham complex of differential forms on the hypersurface complement. One denotes by $P^r\Omega^k_{\mathbb{P}^n} \subset \Omega^k_{\mathbb{P}^n}$ the subspace of forms of order $\mathrm{ord}_X(\omega) \leq k-r+1$, if $k-r+1 \geq 0$, or $P^r\Omega^k = 0$ for $k-r+1 < 0$. The polar filtration $P^\bullet$ is related to the Hodge filtration $F^\bullet$ by $P^r\Omega^m \supset F^r\Omega^m$, by a result of [25].

Proposition 3.4. *Under the generic condition on the external momenta, the forms*

$$\frac{\Omega_\xi}{\Psi_\Gamma^{D/2} V_\Gamma^{k-D\ell/2}} \tag{3.16}$$

span subspaces $P^{r,k}_\xi$ *of the polar filtration* $P^r\Omega^{k-1}_{\mathbb{P}^{n-1}}$ *of a hypersurface complement* $\mathcal{U}(f) \subset \mathbb{P}^{n-1}$*, where*

$$f = \begin{cases} P_\Gamma & k - D(\ell+1)/2 \geq 0 \\ P_\Gamma^{(k-D\ell/2)/m} \Psi_\Gamma^{D/(2m)} & k - D(\ell+1)/2 < 0 < k - D\ell/2, \\ & m = \gcd\{k - D\ell/2, D/2\} \\ \Psi_\Gamma & k - D\ell/2 \leq 0, \end{cases} \tag{3.17}$$

and for the index r *of the filtration in the range*

$$\begin{cases} r \leq D\ell/2 & k - D(\ell+1)/2 \geq 0 \\ r \leq k - \gcd\{k - D\ell/2, D/2\} & k - D(\ell+1)/2 < 0 < k - D\ell/2 \\ r \leq 2k - D(\ell+1)/2 & k - D\ell/2 \leq 0. \end{cases} \tag{3.18}$$

Proof. We are assuming that P_Γ and Ψ_Γ have no common factor, for generic external momenta. Consider first the case where $k - D\ell/2 \geq 0$. This is further divided into two cases: the case where also $k - D(\ell+1)/2 \geq 0$ and the case where $k - D(\ell+1)/2 < 0$. In the first case, the form (3.16) can be written, using (2.31), as

$$\frac{\Delta(\alpha)}{f^m} = \frac{\Psi_\Gamma^{k-D(\ell+1)/2}\Omega_\xi}{P_\Gamma^{k-D\ell/2}}, \tag{3.19}$$

where

$$\alpha = \Psi_\Gamma^{k-D(\ell+1)/2}\omega_\xi \quad \text{and} \quad f = P_\Gamma, \quad \text{with} \quad m = k - D\ell/2. \tag{3.20}$$

Thus, in this case we considered the polar filtration for differential forms on the complement of the projective hypersurface Y_Γ of degree $\ell+1$ defined by $P_\Gamma = 0$. The forms (3.19), for a generic choice of the $(n-k)$-frame ξ, and for varying external momenta p, span a subspace $P^{r,k}_\xi$ of the polar filtration $P^r\Omega^{k-1}_{\mathbb{P}^{n-1}}$, for all $r \leq D\ell/2$. Notice that $r \leq D\ell/2$ also implies $r \leq k$ so that one remains within the nontrivial range $k - r \geq 0$ of the filtration.

In the case where we still have $k - D\ell/2 \geq 0$ but $k - D(\ell+1)/2 < 0$, we let

$$m = \gcd\{k - D\ell/2, D/2\},$$

so that $k - D\ell/2 = n_1 m$ and $D/2 = n_2 m$. We then write (3.16) in the form

$$\frac{\Delta(\alpha)}{f^m} = \frac{\Psi_\Gamma^{k-D\ell/2}\Omega_\xi}{P_\Gamma^{k-D\ell/2}\Psi_\Gamma^{D/2}}, \tag{3.21}$$

with

$$\alpha = \Psi_\Gamma^{k-D\ell/2}\omega_\xi, \quad \text{and} \quad f = P_\Gamma^{n_1}\Psi_\Gamma^{n_2} \quad \text{and} \quad m = \gcd\{k - D\ell/2, D/2\}. \tag{3.22}$$

In this case, we consider the polar filtration associated to the complement of the projective hypersurface defined by the equation $P_\Gamma^{n_1}\Psi_\Gamma^{n_2} = 0$. For a generic choice of the $(n-k)$-frame ξ, and for varying external momenta p, we obtain in this case a subspace $P_\xi^{r,k}$ of the polar filtration $P^r\Omega^{k-1}_{\mathbb{P}^{n-1}}$, for all $r \le k - \gcd\{k - D\ell/2, D/2\}$.

The remaining case is when $k - D\ell/2 < 0$, so that also $k - D(\ell+1)/2 < 0$. In this case, we write (3.16) in the form

$$\frac{\Delta(\alpha)}{f^m} = \frac{P_\Gamma^{-k+D\ell/2}\Omega_\xi}{\Psi_\Gamma^{-k+D(\ell+1)/2}}, \tag{3.23}$$

where

$$\alpha = P_\Gamma^{-k+D\ell/2}\omega_\xi, \quad \text{and} \quad f = \Psi_\Gamma \quad \text{and} \quad m = -k + D(\ell+1)/2. \tag{3.24}$$

We are considering here the polar filtration on forms on the complement of the hypersurface X_Γ defined by $\Psi_\Gamma = 0$. We then obtain, for generic ξ and varying p, a subspace of $P_\xi^{r,k}$ of the filtration $P^r\Omega^{k-1}_{\mathbb{P}^{n-1}}$, for all $r \le 2k - D(\ell+1)/2$. □

3.4. **Milnor fiber.** Suppose then that $k = \operatorname{codim} \operatorname{Sing}(X)$, where $\operatorname{Sing}(X)$ is the singular locus of the hypersurface $X = \{f = 0\}$, with f as in Proposition 3.4 above. In this case, for generic ξ, the linear space Π_ξ cuts the singular locus $\operatorname{Sing}(X)$ transversely and the restriction $X_\xi = X \cap \Pi_\xi$ has isolated singularities.

Recall that, in the case of isolated singularities, there is an isomorphism between the cohomology of the Milnor fiber F_ξ of X_ξ and the total cohomology of the Koszul–deRham complex of forms (2.57) with the total differential $d_f\omega = f\,d\omega - m\,df \wedge \omega$ as above. The explict isomorphism is given by the Poincaré residue map and can be written in the form

$$[\omega] \mapsto [j^*\Delta(\omega_\xi)], \tag{3.25}$$

where $j : F_\xi \hookrightarrow \Pi_\xi$ is the inclusion of the Milnor fiber in the ambient space (see [26], §6).

Let $M(f)$ be the Milnor algebra of f, *i.e.* the quotient of the polynomial ring in the coordinates of the ambient projective space by the ideal of the derivatives of f. When f has isolated singularities, the Milnor algebra is finite dimensional. One denotes by $M(f)_m$ the homogeneous component of degree m of $M(f)$.

It then follows from the identification (3.25) above ([26],§6.2) that, in the case of isolated singularities, a basis for the cohomology $H^r(F_\xi)$ of the Milnor fiber, with $r = \dim \Pi_\xi - 1$ is given by elements of the form

$$\omega_\alpha = \frac{t^\alpha \Delta(\omega_\xi)}{f^m}, \qquad \text{with} \qquad t^\alpha \in M(f)_{m\deg(f)-k}, \tag{3.26}$$

where f is the restriction to Π_ξ of the function of (3.17). We then have the following consequence of Proposition 3.4.

Corollary 3.5. *For a generic $(n-k)$-frame ξ with $n - k = \dim \operatorname{Sing}(X)$, with X the hypersurface of Proposition 3.4, and for a fixed generic choice of the external momenta p under the assumption of Definition 2.4, the Feynman integrand* (3.16) *of* (3.14) *defines a cohomology class in $H^r(F_\xi)$, with $r = \dim \Pi_\xi - 1$ and $F_\xi \subset \Pi_\xi$ the Milnor fiber of the hypersurface with isolated singularities $X_\xi = X \cap \Pi_\xi \subset \Pi_\xi$.*

Proof. By Proposition 3.4, the form (3.16) can be written as

$$\frac{h\Delta(\omega_\xi)}{f^m}, \tag{3.27}$$

where f is as in (3.17), and h is a polynomial of the form

$$h = \begin{cases} \Psi_\Gamma^{k-D(\ell+1)/2} & k - D(\ell+1)/2 \geq 0 \\ \Psi_\Gamma^{k-D\ell/2} & k - D(\ell+1)/2 < 0 < k - D\ell/2 \\ P_\Gamma^{-k+D\ell/2} & k - D\ell/2 \leq 0. \end{cases} \tag{3.28}$$

Let $\mathcal{I}_\xi$ denote the ideal of derivatives of the restriction $f|_{\Pi_\xi}$ of f to Π_ξ. Then let

$$h_\xi = h \mod \mathcal{I}_\xi. \tag{3.29}$$

For a fixed generic choice of the external momenta, this defines an element in the Milnor algebra $M(f|_{\Pi_\xi})$, which lies in the homogeneous component $M(f|_{\Pi_\xi})_{m\deg(f)-k}$, for

$$m = \begin{cases} k - D\ell/2 & k - D(\ell+1)/2 \geq 0 \\ \gcd\{k - D\ell/2, D/2\} & k - D(\ell+1)/2 < 0 < k - D\ell/2 \\ -k + D(\ell+1)/2 & k - D\ell/2 \leq 0. \end{cases} \tag{3.30}$$

Thus, the form (3.27) defines a class in the cohomology $H^r(F_\xi)$ with $r = \dim \Pi_\xi - 1$. □

3.5. **The Feynman integral: slicing.** As in Proposition 2.9, we can reformulate the integral (3.14) in terms of integrals of pullbacks of forms on a hypersurface complement in projective space, using the explicit description of Proposition 3.4 above.

Proposition 3.6. *The integral* (3.14) *can be computed in the form*

$$\mathbb{U}(\Gamma)_\xi = \frac{1}{C(k,D,\ell)} \left(\int_{\partial\Sigma\cap\Pi_\xi} \pi^*(\eta_\xi) + \int_{\Sigma\cap\Pi_\xi} df|_{\Pi_\xi} \wedge \frac{\pi^*(\eta_\xi)}{f|_{\Pi_\xi}} \right), \tag{3.31}$$

where $\pi : \mathbb{A}^n \smallsetminus \{0\} \to \mathbb{P}^{n-1}$ is the projection and η_ξ satisfies

$$\pi^*(\eta_\xi) = \frac{h|_{\Pi_\xi}\Omega_\xi}{(f|_{\Pi_\xi})^m} \tag{3.32}$$

on $\mathbb{A}^n$, where Ω_ξ is given by (3.4) *and f, m and h are as in Proposition 3.4 and Corollary 3.5. The coefficient $C(k,D,\ell)$ is given as in* (2.73).

Proof. As in the case of Proposition 2.9, the result follows by applying Proposition 2.8 and Proposition 3.4, together with the fact that $\Omega_\xi = \Delta(\omega_\xi)$, which can be seen by writing

$$\Omega_\xi = \prod_{i=1}^{n-k} \delta(\langle \xi_i, t\rangle)\, \Omega_n.$$

The coefficient $C(k,D,\ell)$ is given by $C(k,D,\ell) = m\deg(f)$, with m and f as in (3.30) and (3.17). □

4. Oscillatory integrals: Leray and Dimensional Regularizations

A well known method for studying integrals of holomorphic forms on vanishing cycles of a singularity and to relate these to mixed Hodge structures is via oscillatory integrals and their asymptotic expansion (see [4] and Vol.II of [5]). Our main result in this section will be to show that the dimensionally regularized parametric Feynman integrals can be related to the Mellin transform of a Gelfand–Leray form, whose Fourier transform is the oscillatory integral usually considered in the context of singularity theory.

4.1. **Oscillatory integrals and the Gelfand–Leray forms.** We recall briefly some results on oscillatory integrals and their asymptotic expansion. We refer the reader to §2, Vol.II of [5] for more details. In general, an *oscillatory integral* is an expression of the form

$$I(\alpha) = \int_{\mathbb{R}^n} e^{i\alpha f(x)} \phi(x)\, dx_1 \cdots dx_n, \tag{4.1}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ and $\phi : \mathbb{R}^n \to \mathbb{R}$ are smooth functions and $\alpha \in \mathbb{R}^*_+$ is a real parameter. It is well known that, if the phase $f(x)$ is an analytic function in a neighborhood of a critical point x_0, then (4.1) has an asymptotic development for $\alpha \to \infty$ given by a series

$$I(\alpha) \sim e^{i\alpha f(x_0)} \sum_u \sum_{k=0}^{n-1} a_{k,u}(\phi)\, \alpha^u (\log \alpha)^k, \tag{4.2}$$

where u runs over a finite set of arithmetic progressions of negative rational numbers depending only on the phase $f(x)$, and the $a_{k,u}$ are distributions supported on the critical points of the phase, *cf.* §2.6.1, Vol.II of [5].

It is also well known that the integral (4.1) can be reformulated in terms of one-dimensional integrals using the Gelfand–Leray form

$$I(\alpha) = \int_{\mathbb{R}} e^{i\alpha t} \left(\int_{X_t(\mathbb{R})} \phi(x) \omega_f(x,t) \right) \, dt \tag{4.3}$$

where $X_t(\mathbb{R}) \subset \mathbb{R}^n$ is the level set $X_t(\mathbb{R}) = \{x \in \mathbb{R}^n \,:\, f(x) = t\}$ and $\omega_f(x,t)$ is the Gelfand–Leray form, that is, the unique $(n-1)$-form on the level hypersurface X_t with the property that

$$df \wedge \omega_f(x,t) = dx_1 \wedge \cdots \wedge dx_n. \tag{4.4}$$

Notice that, as in the case of the forms (3.4), there is some choice of an $(n-1)$-form satisfying (4.4), but the restriction to X_t is unique so that the Gelfand–Leray form on X_t is well defined. Notice also that, up to throwing away a set of measure zero, we can assume here that the integration is over the values $t \in \mathbb{R}$ such that the level set X_t is a smooth hypersurface.

The Gelfand–Leray form $\omega_f(x,t)$ is often written in the notation

$$\omega_f(x,t) = \frac{dx_1 \wedge \cdots \wedge dx_n}{df}. \tag{4.5}$$

It is given by the Poincaré residue

$$\frac{\omega}{df} = \mathrm{Res}_{\epsilon=0} \frac{\omega}{f-\epsilon}. \tag{4.6}$$

The Gelfand–Leray function is the associated function

$$J(t) := \int_{L_t} \phi(x) \omega_f(x,t). \tag{4.7}$$

For more details, see §2.6 and §2.7, Vol.II [5].

We recall here a property of the Gelfand–Leray forms that will be useful in the following, where we consider complex hypersurfaces $X \subset \mathbb{A}^n = \mathbb{C}^n$, with defining polynomial equation $f = 0$ and the hypersurface complement $\mathcal{D}(f) \subset \mathbb{A}^n$, such that the restriction of f to the interior of the domain of integration $\Sigma \subset \mathbb{A}^n$ takes values in $\mathbb{R}_+^*$.

Recall that the Leray coboundary of a k-chain σ in X is a $(k+1)$-chain in $\mathcal{D}(f)$ obtained by considering a tubular neighborhood of X in $\mathbb{A}^n$, in the following way. Since X is a hypersurface, the boundary of its tubular neighborhood is a circle bundle over X. One considers the preimage of σ under the projection map as a chain in $\mathcal{D}(f)$. We denote the resulting chain by $\mathcal{L}(\sigma)$. It is called the Leray coboundary of σ (see [5] p.282). The Leray coboundary $\mathcal{L}(\sigma)$ is a cycle if σ is a cycle, and if one changes σ by a boundary then $\mathcal{L}(\sigma)$ also changes by a boundary.

Lemma 4.1. *Let σ_ϵ be a k-chain in $X_\epsilon = \{t \in \mathbb{A}^n | f(t) = \epsilon\}$ and let $\mathcal{L}(\sigma_\epsilon)$ be its Leray coboundary in $\mathcal{D}(f - \epsilon)$. Then, for a form $\alpha \in \Omega^k$ that admits a Gelfand–Leray form, one has*

$$\frac{1}{2\pi i} \int_{\mathcal{L}(\sigma(\epsilon))} df \wedge \frac{\alpha}{f - \epsilon} = \int_{\sigma(\epsilon)} \alpha, \tag{4.8}$$

where

$$\frac{d}{d\epsilon} \int_{\sigma(\epsilon)} \alpha = \int_{\sigma(\epsilon)} \frac{d\alpha}{df} - \int_{\partial\sigma(\epsilon)} \frac{\alpha}{df}. \tag{4.9}$$

Proof. First let us show that if α has a Gelfand–Leray form then $d\alpha$ also does. We have a form α/df such that

$$df \wedge \frac{\alpha}{df} = \alpha.$$

Its differential gives

$$d\alpha = d\left(df \wedge \frac{\alpha}{df}\right) = -df \wedge d\left(\frac{\alpha}{df}\right).$$

Thus, the form

$$\frac{d\alpha}{df} = -d\left(\frac{\alpha}{df}\right)$$

is a Gelfand–Leray form for $d\alpha$.

Then we proceed to prove the first statement. One can write

$$\frac{1}{2\pi i} \int_{\mathcal{L}(\sigma(\epsilon))} df \wedge \frac{\alpha}{f - \epsilon} = \frac{1}{2\pi i} \int_\gamma \left(\int_{\sigma(s)} \alpha \right) \frac{ds}{s - \epsilon},$$

where $\gamma \cong S^1$ is the boundary of a small disk centered at $\epsilon \in \mathbb{C}$. This can then be written as

$$= \frac{1}{2\pi i} \int_\gamma \int_{\sigma(\epsilon)} \alpha \frac{ds}{s - \epsilon} + \left(\frac{1}{2\pi i} \int_\gamma \int_{\sigma(s)} \alpha \frac{ds}{s - \epsilon} - \frac{1}{2\pi i} \int_\gamma \int_{\sigma(\epsilon)} \alpha \frac{ds}{s - \epsilon} \right).$$

The last term can be made arbitrarily small, so one gets (4.8). To obtain (4.9) notice that

$$\frac{1}{2\pi i} \frac{d}{d\epsilon} \int_{\mathcal{L}(\sigma(\epsilon))} df \wedge \frac{\alpha}{f - \epsilon} = \frac{1}{2\pi i} \int_{\mathcal{L}(\sigma(\epsilon))} df \wedge \frac{\alpha}{(f - \epsilon)^2}.$$

One then uses

$$d\left(\frac{\alpha}{f - \epsilon}\right) = \frac{d\alpha}{f - \epsilon} - \frac{\alpha}{(f - \epsilon)^2}$$

to rewrite the above as

$$\frac{1}{2\pi i} \left(\int_{\mathcal{L}(\sigma(\epsilon))} \frac{d\alpha}{f - \epsilon} - \int_{\mathcal{L}(\sigma(\epsilon))} d\left(\frac{\alpha}{f - \epsilon}\right) \right)$$

$$= \frac{1}{2\pi i} \int_{\mathcal{L}(\sigma(\epsilon))} \frac{df \wedge \frac{d\alpha}{df}}{f-\epsilon} - \frac{1}{2\pi i} \int_{\mathcal{L}(\partial\sigma(\epsilon))} \frac{\alpha}{f-\epsilon}$$

$$= \frac{1}{2\pi i} \int_{\mathcal{L}(\sigma(\epsilon))} \frac{df \wedge \frac{d\alpha}{df}}{f-\epsilon} - \frac{1}{2\pi i} \int_{\mathcal{L}(\partial\sigma(\epsilon))} \frac{df \wedge \frac{\alpha}{df}}{f-\epsilon},$$

where $d\alpha/df$ is a Gelfand–Leray form such that

$$df \wedge \frac{d\alpha}{df} = d\alpha,$$

and α/df is a Gelfand–Leray form with the property that

$$df \wedge \frac{\alpha}{df} = \alpha.$$

This then gives by (4.8)

$$\frac{d}{d\epsilon} \int_{\sigma(\epsilon)} \alpha = \int_{\sigma(\epsilon)} \frac{d\alpha}{df} - \int_{\partial\sigma(\epsilon)} \frac{\alpha}{df}.$$

This completes the proof. □

4.2. **Leray coboundary regularization and subtraction.** The formulation (2.68) of the parametric Feynman integrals, in the form of Proposition 2.9, suggests a regularization procedure different from Dimensional Regularization, but with the similar effect of replacing a divergent integral with a meromorphic function to which the "minimal subtraction" procedure can be applied to remove the polar part and extract a finite value.

Since the singularities arise where the domain of integration Σ meets the hypersurface $X = \{f = 0\}$, with f as in (2.70), we can concentrate on only the part of the integral that is supported near this intersection.

Let $D_\epsilon(X)$ denote a neighborhood of the hypersurface X in $\mathbb{P}^{n-1}$, given by level sets

$$D_\epsilon(X) = \cup_{s\in\Delta^*_\epsilon} X_s, \tag{4.10}$$

where $X_s = \{t | f(t) = s\}$ and $\Delta^*_\epsilon \subset \mathbb{C}^*$ is a small punctured disk of radius $\epsilon > 0$. The boundary $\partial D_\epsilon(X)$ is given by

$$\partial D_\epsilon(X) = \cup_{s\in\partial\Delta^*_\epsilon} X_s. \tag{4.11}$$

It is a circle bundle over the generic fiber X_ϵ, with projection $\pi_\epsilon : \partial D_\epsilon(X) \to X_\epsilon$. Given a domain of integration Σ, we consider the intersection $\Sigma \cap D_\epsilon(X)$. This is the region that contains the locus $\Sigma \cap X$ where the divergence in the Feynman integral can occur. We let $\mathcal{L}_\epsilon(\Sigma)$ denote the set

$$\mathcal{L}_\epsilon(\Sigma) = \pi_\epsilon^{-1}(\Sigma \cap X_\epsilon). \tag{4.12}$$

This enjoys the same properties of the Leray coboundary discussed above. In particular, notice that $\mathcal{L}_\epsilon(\partial\Sigma) = \partial\mathcal{L}_\epsilon(\Sigma)$.

We consider forms $\pi^*(\eta)$ as in (2.69). To keep track explicitly of the order of pole of such forms along the hypersurface X, we modify the notation and write

$$\pi^*(\eta_m) = \frac{\Delta(\omega)}{f^m}, \tag{4.13}$$

with ω and f as in (2.69).

We then make the following proposal for a regularization method for the Feynman integrals (2.68). We call it *Leray regularization*, because it is based on the use of Leray coboundaries. (Notice that this procedure of regularization and subtraction happens after having already removed the divergent Γ-factor from the parametric Feyman integrals and

passing to residues. It is meant in fact to take care of the remaining singularities that arise from the intersections of the hypersurface with the domain of integration.)

Definition 4.2. *The Leray regularized Feynman integral is obtained from* (2.68) *by replacing the part*

$$\int_{\partial\Sigma\cap D_\epsilon(X)} \pi^*(\eta_m) + \int_{\Sigma\cap D_\epsilon(X)} df \wedge \frac{\pi^*(\eta_m)}{f} \tag{4.14}$$

of (2.68) *with the integral*

$$\int_{\mathcal{L}_\epsilon(\partial\Sigma)} \frac{\pi^*(\eta_{m-1})}{f-\epsilon} + \int_{\mathcal{L}_\epsilon(\Sigma)} df \wedge \frac{\pi^*(\eta_m)}{f-\epsilon}. \tag{4.15}$$

Thus, the Leray regularization introduced here consists of replacing the integral over $\Sigma \cap D_\epsilon(X)$ with an integral over $\mathcal{L}_\epsilon(\Sigma) \simeq (\Sigma \cap X_\epsilon) \times S^1$, which avoids the locus $\Sigma \cap X$ where the divergence can occur by going around it along a circle of small radius $\epsilon > 0$.

Using the result of Lemma 4.1, we can formulate (4.15) equivalently in the following form.

Lemma 4.3. *The Leray regularization of the Feynman integral* (2.68) *can be equivalently written in the form*

$$\begin{aligned} \mathbb{U}(\Gamma)_\epsilon = \quad & \frac{1}{C(n,D,\ell)} \left(\int_{\partial\Sigma\cap D_\epsilon(X)^c} \pi^*(\eta_m) + \int_{\Sigma\cap D_\epsilon(X)^c} df \wedge \frac{\pi^*(\eta_m)}{f} \right) \\ + \quad & \frac{2\pi i}{C(n,D,\ell)} \left(\int_{\partial\Sigma\cap X_\epsilon} \frac{\pi^*(\eta_{m-1})}{df} + \int_{\Sigma\cap X_\epsilon} \pi^*(\eta_m), \right) \end{aligned} \tag{4.16}$$

with $\pi^*(\eta_m) = \Delta(\omega)/f^m$ *as in* (4.13) *and Proposition 2.9.*

Proof. The result follows directly from Proposition 2.9 and Lemma 4.1 applied to (4.15). ∎

In (4.16) we use the notation $D_\epsilon(X)^c$ to denote the complement of $D_\epsilon(X)$. Notice how only the part of the integral (2.68) that is computed inside $D_\epsilon(X)$ is replaced by (4.15) in the Leray regularization, while the part of the integral (2.68) computed outside of $D_\epsilon(X)$ remains unchanged.

We now study the dependence on the parameter $\epsilon > 0$ of the Leray regularized Feynman integral (4.15), that is, of the integral

$$I_\epsilon := \int_{\partial\Sigma\cap X_\epsilon} \frac{\pi^*(\eta_{m-1})}{df} + \int_{\Sigma\cap X_\epsilon} \pi^*(\eta_m). \tag{4.17}$$

Theorem 4.4. *The function* I_ϵ *of* (4.17) *is infinitely differentiable in* ϵ*. Moreover, it extends to a holomorphic function for* $\epsilon \in \Delta^* \subset \mathbb{C}$*, a small punctured disk, with a pole of order at most* m *at* $\epsilon = 0$*, with* m *as in* (2.71).

Proof. To prove the differentiability of I_ϵ, let us write

$$A_\epsilon(\eta) = \int_{\Sigma\cap X_\epsilon} \pi^*(\eta), \tag{4.18}$$

with $\pi^*(\eta)$ as in (2.69). By Lemma 4.1 above, and the fact that $d\pi^*(\eta) = 0$, we obtain

$$\frac{d}{d\epsilon} A_\epsilon(\eta) = -\int_{\partial\Sigma\cap X_\epsilon} \frac{\pi^*(\eta)}{df}, \tag{4.19}$$

where f is as in (2.70) and $\pi^*(\eta)/df$ is the Gelfand–Leray form of $\pi^*(\eta)$. Thus, we can write

$$I_\epsilon = A_\epsilon(\eta_m) - \frac{d}{d\epsilon} A_\epsilon(\eta_{m-1}).$$

Thus, to check the differentiability in the variable ϵ to all orders of I_ϵ is equivalent to checking that of A_ϵ. We define then $\Upsilon : \Omega^n \to \Omega^n$ by setting

$$\Upsilon(\alpha) = d\left(\frac{\alpha}{df}\right), \tag{4.20}$$

where α/df is a Gelfand–Leray form for α. In turn, the n-form $\Upsilon(\alpha)$ also has a Gelfand–Leray form, which we denote by

$$\delta(\alpha) = \frac{\Upsilon(\alpha)}{df} = \frac{d\left(\frac{\alpha}{df}\right)}{df}. \tag{4.21}$$

We then prove that, for $k \geq 2$,

$$\frac{d^k}{d\epsilon^k} A_\epsilon = -\int_{\partial\Sigma\cap X_\epsilon} \delta^{k-1}\left(\frac{\pi^*(\eta)}{df}\right). \tag{4.22}$$

This follows by induction. In fact, we first see that

$$\frac{d^2}{d\epsilon^2} A_\epsilon = -\frac{d}{d\epsilon}\int_{\partial\Sigma\cap X_\epsilon} \frac{\pi^*(\eta)}{df}$$

which, applying Lemma 4.1 gives

$$= -\int_{\partial\Sigma\cap X_\epsilon} \frac{d\left(\frac{\pi^*(\eta)}{df}\right)}{df}.$$

Assuming then that

$$\frac{d^k}{d\epsilon^k} A_\epsilon = -\int_{\partial\Sigma\cap X_\epsilon} \delta^{k-1}\left(\frac{\pi^*(\eta)}{df}\right)$$

we obtain again by a direct application of Lemma 4.1

$$\frac{d^{k+1}}{d\epsilon^{k+1}} A_\epsilon = -\int_{\partial\Sigma\cap X_\epsilon} \frac{d\left(\frac{\delta^{k-1}\left(\frac{\pi^*(\eta)}{df}\right)}{df}\right)}{df}$$
$$= -\int_{\partial\Sigma\cap X_\epsilon} \delta^{k}\left(\frac{\pi^*(\eta)}{df}\right).$$

This proves differentiability to all orders.

Notice then that, while the expression (4.15) used in Definition 4.2 is, a priori, only defined for $\epsilon > 0$, the equivalent expression given in the second line of (4.16) and in (4.17) is clearly defined for any complex $\epsilon \in \Delta^*$ in a punctured disk around $\epsilon = 0$ of sufficiently small radius. It can then be seen that the expression (4.17) depends holomorphically on the parameter ϵ by the general argument on holomorphic dependence on parameters given in Part III, §10.2 of Vol.II of [5].

Finally, to see that I_ϵ has a pole of order at most m at $\epsilon = 0$, notice that the form $\pi^*(\eta_m)$ of (4.13) is given by $\Delta(\omega)/f^m$ and has a pole of order at most m at X. This is evident in the two cases with $n - \frac{D(\ell+1)}{2} \geq 0$ or $n - \frac{D\ell}{2} \leq 0$. It also holds in the intermediate case with $n - \frac{D(\ell+1)}{2} < 0 < n - \frac{D\ell}{2}$, since we are taking the convention that, in the case a hypersurface X defined by a polynomial $f = f_1^{n_1} f_2^{n_2}$, a form $\Delta(\omega)/f^m$ has pole order m along X, even thought on the individual components it has order mn_1 and mn_2, respectively. $\square$

In particular, the result of Proposition 4.4 shows that we can use the Leray regularization as an alternative to dimensional regularization to replace a divergent Feynman integral by a meromorphic function of a complex variable ϵ with a pole at $\epsilon = 0$. It is then possible to proceed as in dimensional regularization and apply "minimal subtraction", namely subtract the polar part of the resulting Laurent series in ϵ and evaluate the remaining part at $\epsilon = 0$.

It is clear that this regularization method is subject to the same problems as dimensional regularization when it comes to considering Feynman integrals associated to graphs that contain subdivergences. One can organize the hierarchy of subdivergences using the Bogolyubov-Parashuk preparation, as in the case of dimensional regularization.

4.3. **Birkhoff factorization and renormalization.** Connes and Kreimer [21] showed that the BPHZ renormalization procedure, in the DimReg+MS regularization scheme, can be understood conceptually as the Birkhoff factorization of loops in the Lie group of complex points of the affine group scheme G dual to a commutative Hopf algebra $\mathcal{H}$ generated by the Feynman diagrams of the given physical theory. The Hopf algebra $\mathcal{H}$, at the discrete combinatorial level, is the commutative algebra generated by the one-particle-irreducible (1PI) graphs of the theory, with the coproduct

$$\Delta(\Gamma) = \Gamma \otimes 1 + 1 \otimes \Gamma + \sum \gamma \otimes \Gamma/\gamma, \tag{4.23}$$

where the sum is over proper subgraphs $\gamma \subset \Gamma$ satisfying a set of properties such as being Feynman diagrams of the same theory (see for instance [23] for a detailed discussion of the assumptions on the family of subgraphs involved in the coproduct). The quotient Γ/γ denotes the graph obtained by contracting each component of γ to a single vertex. It is sometimes denoted in the literature with the notation $\Gamma//\gamma$. The Hopf algebra is graded by the number of internal lines of graphs.

After identifying loops $\gamma : \Delta^* \to G(\mathbb{C})$, defined on an infinitesimal punctured disk Δ^* around $z = 0$, with elements $\phi \in G(K) = \mathrm{Hom}(\mathcal{H}, K)$, where K is the field of germs of meromorphic functions at $z = 0$, Connes and Kreimer showed that the BPHZ formula for renormalization is the recursive formula

$$\begin{aligned} \phi_-(x) &= -T(\phi(x) + \textstyle\sum \phi_-(x')\phi(x'')) \\ \phi_+(x) &= \phi(x) + \phi_-(x) + \textstyle\sum \phi_-(x')\phi(x''), \end{aligned} \tag{4.24}$$

with $\Delta(x) = x \otimes 1 + 1 \otimes x + \sum x' \otimes x''$, and x', x'' of lower degree, and with T the projection of a Laurent series onto its polar part. The original BPHZ formula is obtained by applying (4.24) to the element $\phi \in \mathrm{Hom}(\mathcal{H}, K)$ that assigns to a generator Γ of $\mathcal{H}$ its unrenormalized Feynman integral $U(\Gamma)$. As shown in [21], the formula (4.24) is the recursive formula that gives the Birkhoff factorization

$$\gamma(z) = \gamma_-(z)^{-1}\gamma_+(z) \tag{4.25}$$

of the loop γ into a part γ_+ that is holomorphic on Δ and a part γ_- that is holomorphic at $\infty \in \mathbb{P}^1(\mathbb{C})$, where one identifies γ_+ with $\phi_+ \in \mathrm{Hom}(\mathcal{H}, \mathcal{O})$, with $\mathcal{O}$ the algebra of germs of holomorphic functions at $z = 0$ and γ_- with $\phi_- \in \mathrm{Hom}(\mathcal{H}, \mathcal{Q})$ wth $\mathcal{Q} = \mathbb{C}[z^{-1}]$ so that

$$\phi = (\phi_- \circ S) * \phi_+, \tag{4.26}$$

with S the antipode of $\mathcal{H}$ and $*$ the product in the affine group scheme G, dual to the coproduct of $\mathcal{H}$.

The formulation in terms of Birkhoff factorization of loops with values in the Lie group of complex points of the affine group scheme of diffeographisms is applied in [21] to the Dimensional Regularization of Feynman integrals. Namely, the dimensionally regularized Feynman integrals $U(\Gamma)(z)$ of (2.43) define an element $\phi \in \mathrm{Hom}(\mathcal{H}, K)$, with $\mathcal{H}$ the

Connes–Kreimer Hopf algebra of Feynman graphs of the theory and K the field of germs of meromorphic functions at $z = 0$, given by assigning as values on the generators of the Hopf algebra

$$\phi(\Gamma) = U(\Gamma) \in K. \tag{4.27}$$

In the case of dimensional regularization of Feynman integrals, the fact that the $U(\Gamma)(z)$ define meromorphic functions is very delicate, see the discussion in §1.4 of [23], especially Lemma 1.7, Lemma 1.8, and Theorem 1.9. On the contrary, we have seen that, using the Leray coboundary regularization introduced above, one easily obtains meromorphic functions of the parameter ϵ. We return to discuss the analytic continuation to meromorphic functions of the dimensionally regularized integrals via a different approach in §4.4 below.

By the results of §4.2 above, we can apply the same BPHZ renormalization procedure to the Leray coboundary regularization introduced in Definition 4.3. We thus consider the element $\phi \in \mathrm{Hom}(\mathcal{H}, K)$ defined by assigning on generators

$$\phi(\Gamma)(\epsilon) = U(\Gamma)_\epsilon \tag{4.28}$$

defined as in (4.16). By Proposition 4.4, we know that $U(\Gamma)_\epsilon$ defines a germ of a meromorphic function for $\epsilon \in \Delta^*$, an infinitesimal punctured disk around $\epsilon = 0$, hence it defines an element in K. We can then apply the Birkhoff factorization of ϕ, as in (4.26). This provides the counterterms, in the form

$$C(\Gamma)_\epsilon = \phi_-(\Gamma)(\epsilon), \tag{4.29}$$

which, as a function of ϵ, is an element in $\mathcal{Q}$, and the renormalized value of the Feynman integral, given by the finite value at zero

$$R(\Gamma) = \phi_+(\Gamma)(0), \tag{4.30}$$

where $\phi_+(\Gamma)(\epsilon)$ defines an element in the ring of convergent power series $\mathcal{O} \subset K$.

4.4. **Mellin transform and the DimReg integral.** We now return to consider the method of Dimensional Regularization and reinterpret it in terms of oscillatory integrals and mixed Hodge structures. As we recalled briefly in §4.1 above, the oscillatory integrals used in the theory of singularities and mixed Hodge structures can be seen as Fourier transforms (4.3) of a Gelfand–Leray function (4.7). One can also consider, instead of a Fourier transform, a Mellin transform of the same Gelfand–Leray function. Since Mellin and Fourier transform determine each other by well known formulae, the information obtained in this way is equivalent. In the context of singularity theory, the Mellin transforms of Gelfand–Leray functions and its relation to the oscillatory integral is discussed, for instance, in Part II, §7.2.1, of [5], Vol.II.

It was already proved by Belkale and Brosnan in [9] that, in the case of log-divergent graphs, the dimensionally regularized parametric Feynman integral can be written as a local Igusa L-function. This was later generalized to the non-log-divergent case in the work of Bogner and Weinzierl [16], [17], [18]. Our approach here is closely related to these results, though we do not discuss in detail the explicit relation. Moreover, we simplify the form of the integrals with respect to the case considered by Bogner and Weinzierl, so that we do not have to perform the cutting into sectors and blowups. We rely, in fact, on the formulation in terms of the exponential of the rational function $V_\Gamma(t, p)$ and its expansion, and we analyze the resulting terms individually. A more detailed analysis using the formulation of Bogner–Weinzierl and Belkale–Brosnan is possible, but we do not consider it here.

In order to relate the dimensionally regularized parametric Feynman integral to the oscillatory integrals and the Mellin transforms of Gelfand–Leray functions, consider again

the integrals of the form (2.43), or better, the similar integrals computed after slicing with a k-plane Π_ξ as in §3.5, so that the intersection $X_\Gamma \cap \Pi_\xi$ has isolated singularities.

As shown in Lemma 2.6, we can equivalently compute the dimensionally regularized Feynman integral (2.43) using the form (2.44). Thus, we first consider an integral of the form

$$\int_{\Pi_\xi^+} \frac{e^{-V_\Gamma(t,p)}}{\Psi_\Gamma(t)^{(D+z)/2}}\, \omega_\xi \tag{4.31}$$

where $\Pi_\xi^+ = \Pi_\xi \cap \mathbb{R}^n_+$ and ω_ξ is as in (3.11). After expanding the exponential term and using (2.31), we are reduced to considering integrals of the form

$$\int_{\Pi_\xi^+} \frac{P_\Gamma(t,p)^\ell}{\Psi_\Gamma(t)^{\ell+(D+z)/2}}\, \omega_\xi. \tag{4.32}$$

Thus, we concentrate here on integrals of the form

$$F_{\Gamma,\xi}(z) = \int \Psi_\Gamma^z\, \chi_\xi\, P_\Gamma^\ell \Omega_\xi, \tag{4.33}$$

with Ω_ξ is as in (3.4), and for some integer $\ell \geq 0$. We have made here a simple change of coordinates on the complex variable z, whose meaning will become apparent in a moment.

The function χ_ξ in (4.33) is the characteristic function of the domain of integration. In order to show that one can extract from these dimensionally regularized Feynman integrals information on the singularities of the graph hypersurface X_Γ (through its slices $X_\Gamma \cap \Pi_\xi$), it suffices to concentrate on the part of the domain of integration that is close to the hypersurface X_Γ. Thus, we can include in the function χ_ξ an additional cutoff of the integral that is supported in a neighborhood of the intersection $\Sigma_\xi \cap X_\Gamma$ of the original domain of integration in Π_ξ with the graph hypersurface.

In the following, for simplicity of notation, we just write (4.33) as

$$F_{\Gamma,\xi}(z) = \int \Psi_\Gamma^z\, \alpha_\xi, \tag{4.34}$$

where

$$\alpha_\xi = \chi_\xi\, P_\Gamma^\ell\, \Omega_\xi. \tag{4.35}$$

Lemma 4.5. *The function* (4.34) *is the Mellin transform of the Gelfand-Leray function*

$$J_{\Gamma,\xi}(\epsilon) = \int_{X_\epsilon} \frac{\alpha_\xi}{df}, \tag{4.36}$$

with $f = \Psi_\Gamma|_{\Pi_\xi}$.

Proof. First observe that both functions $\Psi_\Gamma(t)$ and $P_\Gamma(t,p)$ are real when restricted to the domain $\Sigma \subset \mathbb{R}^n_+$, with $\Psi_\Gamma(t) > 0$ on the interior of this domain. Thus, we can write the function $F_{\Gamma,\xi}(z)$ of (4.34) in the form

$$F_{\Gamma,\xi}(z) = \int_0^\infty s^z \left(\int_{X_s} \frac{\alpha_\xi}{df} \right) ds. \tag{4.37}$$

One can recognize then that (4.37) is in fact the Mellin transform

$$F_{\Gamma,\xi}(z) = \int_0^\infty s^z\, J_{\Gamma,\xi}(s)\, ds, \tag{4.38}$$

for $J_{\Gamma,\xi}$ as in (4.36), the corresponding Gelfand–Leray function. $\square$

The identification of $F_{\Gamma,\xi}(z)$ with the Mellin transform (4.38) also provides an answer to the problem of the analytic continuation to meromorphic functions in the complex plane for functions of the form (4.33). This analytic continuation is needed in order to justify our change of variables in z in passing from (4.32) to (4.33), as well as the use of integrals of the form (4.33) to derive conclusions about the original dimensionally regularized integrals (4.32). In fact, the existence of an analytic continuation to meromorphic functions for the functions $F_{\Gamma,\xi}(z)$ follows from the existence of an asymptotic expansion for Gelfand–Leray functions of the form

$$J(s) = \int_{X_s} \frac{\alpha}{df}, \quad \alpha = h\chi\omega_n, \tag{4.39}$$

with h a polynomial term and χ a compactly supported smooth function, supported near an isolated singularity of the hypersurface $f = 0$. The asymptotic expansion is given by

$$J(s) \sim \sum_{\lambda \in \Xi} \sum_{r=0}^{n-1} a_{r,\lambda}\, s^{\lambda} \log(s)^r, \qquad s \to 0^+ \tag{4.40}$$

with Ξ a discrete subset of $\mathbb{R}$. The points $\lambda \in \Xi$ depend on the set of multiplicities of an embedded resolution of the singularity, see Part II, §7 of [5] and [43]. This implies the following result (*cf.* [5]), for generic choice of the slicing Π_ξ and of the external momenta.

Corollary 4.6. *Suppose that the cutoff function χ_ξ in* (4.35) *is supported in a small neighborhood of an isolated singularity of $X_\Gamma \cap \Pi_\xi$. Then the function $F_{\Gamma,\xi}(z)$, defined as in* (4.34) *for $\Re(z) > 0$ sufficiently large, admits an analytic continuation to a meromorphic function over the whole complex plane, with poles at the discrete set of points $z = -(\lambda+1)$, with $\lambda \in \Xi$ as in* (4.40)*, with the coefficient of $(z+\lambda+1)^{-(r+1)}$ in the Laurent series expansion given by $(-1)^r r! a_{r,\lambda}$, with $a_{r,\lambda}$ as in* (4.40)*.*

4.5. **Dimensional regularization and mixed Hodge structures.** We use the results of the previous section relating the dimensional regularization of the Feynman integrals to the Mellin transform of Gelfand–Leray functions, and the results of §3.4 on the interpretation in terms of cohomology of the Milnor fiber, to relate the dimensionally regularized Feynman integrals to limiting mixed Hodge structures.

We assume here to be in the case of isolated singularities, possibly after replacing the original Feynman integrals with their slices along planes Π_ξ of dimension complementary to that of the singular locus of the hypersurface, as discussed in §§3.2 and 3.5 above.

The cohomological Milnor fibration has fiber over ϵ given by the complex vector space $H^{k-1}(F_\epsilon, \mathbb{C})$, where the Milnor fiber F_ϵ of X_ξ is homotopically a bouquet of μ spheres S^{k-1}, with $k = \dim \Pi_\xi - 1$ and with μ the Milnor number of the isolated singularity. A holomorphic k form $\alpha = h\omega_\xi / f^m$ determines a section of the cohomological Milnor fibration by taking the classes

$$\left[\frac{\alpha}{df}|_{F_\epsilon} \right] \in H^{k-1}(F_\epsilon, \mathbb{C}). \tag{4.41}$$

We then have the following results ([5], Vol.II §13). The asymptotic formula (4.40) for the Gelfand–Leray functions implies that the function of ϵ obtained by pairing the section (4.41) with a locally constant section of the homological Milnor fibration has an asymptotic expansion

$$\left\langle \left[\frac{\alpha}{df} \right], \delta \right\rangle \sim \sum_{\lambda, r} \frac{a_{r,\lambda}}{r!} \epsilon^{\lambda} \log(\epsilon)^r, \tag{4.42}$$

for $\epsilon \to 0$, where $\delta(\epsilon) \in H_{k-1}(F_\epsilon, \mathbb{Z})$. Moreover, there exist classes

$$\eta^\alpha_{r,\lambda}(\epsilon) \in H^{k-1}(F_\epsilon, \mathbb{C}) \tag{4.43}$$

such that the coefficients $a_{r,\lambda}$ of (4.42) are given by

$$\langle \eta^\alpha_{r,\lambda}(\epsilon), \delta(\epsilon)\rangle = a_{r,\lambda}. \tag{4.44}$$

Thus, one defines the "geometric section" associated to the holomorphic k-form α as

$$\sigma(\alpha) := \sum_{r,\lambda} \eta^\alpha_{r,\lambda}(\epsilon) \frac{\epsilon^\lambda \log(\epsilon)^r}{r!}. \tag{4.45}$$

The order of the geometric section $\sigma(\alpha)$ is defined as being the smallest λ in the discrete set $\Xi \subset \mathbb{R}$ such that $\eta^\alpha_{0,\lambda} \neq 0$. One denotes it with λ_α. The *principal part* of $\sigma(\alpha)$ is then defined as

$$\sigma_{\max}(\alpha)(\epsilon) := \epsilon^{\lambda_\alpha} \left(\eta^\alpha_{0,\lambda_\alpha} + \cdots + \frac{\log(\epsilon)^{k-1}}{(k-1)!} \eta^\alpha_{k-1,\lambda_\alpha} \right), \tag{4.46}$$

where one knows that

$$\eta^\alpha_{r,\lambda} = \mathcal{N}^r \eta^\alpha_{0,\lambda}, \tag{4.47}$$

where $\mathcal{N}$ is the nilpotent operator given by the logarithm of the unipotent monodromy, given by

$$\mathcal{N} = -\frac{1}{2\pi i} \log \mathcal{T}$$

with $\log \mathcal{T} = \sum_{r\geq 1} (-1)^{r+1} (\mathcal{T} - id)^r / r$.

The *asymptotic mixed Hodge structure* on the fibers of the cohomological Milnor fibration constructed by Varchenko ([44], [45]) has as the Hodge filtration the subspaces $F^r \subset H^{k-1}(F_\epsilon, \mathbb{C})$ defined by

$$F^r = \{ [\alpha/df] \,|\, \lambda_\alpha \leq k - r - 1 \} \tag{4.48}$$

and as weight filtration $W_\ell \subset H^{k-1}(F_\epsilon, \mathbb{C})$ the filtration associated to the nilpotent monodromy operator $\mathcal{N}$. This mixed Hodge structure has the same weight filtration as the *limiting mixed Hodge structure* constructed by Steenbrink ([39], [40]), but the Hodge filtration is different, though the two agree on the graded pieces of the weight filtration.

We now use a refined version of the results of §3.4, and in particular Corollary 3.5 for Feynman integrands as in (3.16). We show that, upon varying the choice of the external momenta p and of the spacetime dimension D, the corresponding Feynman integrands, in a neighborhood of an isolated singular point of $X_\Gamma \cap \Pi_\xi$, determine a subspace of the cohomology $H^{k-1}(F_\xi, \mathbb{C})$ of the Milnor fiber of $X_\Gamma \cap \Pi_\xi$. This inherits a Hodge and a weight filtrations from the Milnor fiber cohomology with its asymptotic mixed Hodge structure. We concentrate on the case where $k - D\ell/2 \leq 0$, so that we can consider, for fixed k, arbitrarily large values of $D \in \mathbb{N}$.

Proposition 4.7. *Consider Feynman integrals, sliced along a linear space* Π_ξ *as in* (3.14). *We write the integrand in the form*

$$\alpha_\xi = \frac{h\Omega_\xi}{f^m}, \tag{4.49}$$

with

$$\begin{cases} h = P_\Gamma^{-k+D\ell/2} \\ f = \Psi_\Gamma \\ m = -k + D(\ell+1)/2, \end{cases} \tag{4.50}$$

as in (3.24)*, with* $k - D\ell/2 \leq 0$*. Upon varying the external momenta* p *in* $P_\Gamma(p,t)$ *and the spacetime dimension* $D \in \mathbb{N}$*, with* $k - D\ell/2 \leq 0$*, the forms* α_ξ *as above determine a subspace*

$$H^{k-1}_{\rm Feynman}(F_\epsilon,\mathbb{C}) \subset H^{k-1}(F_\epsilon,\mathbb{C}),$$

of the fibers of the cohomological Milnor fibration, spanned by elements of the form (4.49)*, where the polynomials* $h = h_{T,v,w,p}$ *are of the form*

$$h(t) = \prod_{i=1}^{-k+D\ell/2} L_{T_i}(t) \prod_{e\notin T_i} t_e, \tag{4.51}$$

where the T_i *are spanning trees and the* $L_{T_i}(t)$ *are the linear functions of* (2.37)*.*

Proof. Consider the explicit expression (2.32) of the polynomial $P_\Gamma(t,p)$ as a function of the external momenta, through the coefficients s_C of (2.33). One can see that, by varying arbitrarily the external momenta, subject to the global conservation law (2.35), one can reduce to the simplest possible case, where all external momenta are zero except for a pair of opposite momenta $P_{v_1} = p = -P_{v_2}$ associated to a pair of external edges attached to a pair of vertices v_1, v_2. In such a case, the polynomial $P_\Gamma(t,p)$ becomes of the form (2.36). Thus, when considering powers $P_\Gamma(t,p)^{-k+D\ell/2}$ for varying D, we obtain all polynomials of the form (4.51). $\square$

We denote by $H^{k-1}_{\rm Feynman}(F_\epsilon,\mathbb{C})$ the subspace of the cohomology $H^{k-1}(F_\epsilon,\mathbb{C})$ of the Milnor fiber spanned by the classes $[\alpha_\xi/df]$ with α_ξ of the form (4.49), with h of the form (4.51), considered modulo the ideal generated by the derivatives of $f=\Psi_\Gamma$ and localized at an isolated singular point, *i.e.* viewed as elements in the Milnor algebra $M(f)$. The subspace $H^{k-1}_{\rm Feynman}(F_\epsilon,\mathbb{C})$ inherits a Hodge and a weight filtration $F^\bullet \cap H^{k-1}_{\rm Feynman}$ and $W_\bullet \cap H^{k-1}_{\rm Feynman}$ from the asymptotic mixed Hodge structure of Varchenko on $H^{k-1}(F_\epsilon,\mathbb{C})$. It is an interesting problem to see whether the subspace $H^{k-1}_{\rm Feynman}$ recovers the full $H^{k-1}(F_\epsilon,\mathbb{C})$ and if $(F^\bullet\cap H^{k-1}_{\rm Feynman}, W_\bullet \cap H^{k-1}_{\rm Feynman})$ still give a mixed Hodge structure, at least for some classes of graphs Γ.

5. Regular and irregular singular connections

An important and still mysterious aspect of the motivic approach to Feynman integrals and renormalization is the problem of reconciling the Riemann–Hilbert correspondence of perturbative renormalization formulated by Connes–Marcolli in [22] (see also [23]), which is based on equivalence classes of certain *irregular singular* connections, with the setting of motives (especially mixed Tate motives) and mixed Hodge structures, which are naturally related to *regular singular* connections. The irregular singular connections of [22] have values in the Lie algebra of the Connes–Kreimer group of diffeographisms and are defined on a fibration over a punctured disk with fiber the multiplicative group, respectively representing the complex variable z of dimensional regularization and the energy scale μ (or rather μ^z) upon which the dimensionally regularized Feynman integrals depend. On the other hand, in the case of hypersurfaces in projective spaces, the natural associated regular singular connection is the Gauss–Manin connection on the cohomology of the Milnor fiber and the Picard–Fuchs equation for the vanishing cycles. We sketch here a relation between this regular singular connection and the irregular equisingular connections of [22]. (To avoid any possible confusion, the reader should keep in mind that the use of the term "equisingular" in [22] is not the same as the well established use in singularity theory, as in [42] for instance.)

5.1. **Picard–Fuchs equation and Gauss–Manin connection.** In the following we let

$$\left[\frac{\omega_i}{df}\right] \qquad i = 1, \dots, \mu \tag{5.1}$$

be a basis for the vanishing cohomology bundle, written with the same notation we used above for the Gelfand–Leray form. Then the Gauss–Manin connection on the vanishing cohomology bundle, which is defined by the integer cohomology lattice in each real cohomology fiber, acts on the basis (5.1) by

$$\nabla_s^{GM} \left[\frac{\omega_i}{df}\right]_s = \sum_j p_{ij}(s) \left[\frac{\omega_j}{df}\right]_s, \tag{5.2}$$

where the $p_{ij}(s)$ are holomorphic away from $s = 0$ and have a pole at $s = 0$. The Gauss–Manin connection is regular singular and its monodromy agrees with the monodromy of the singularity (see [4], §2.3). Given a covariantly constant section $\delta(s)$ of the vanishing homology bundle, the function

$$I(s) = \left(\int_{\delta(s)} \frac{\omega_1}{df}, \dots, \int_{\delta(s)} \frac{\omega_\mu}{df} \right) \tag{5.3}$$

is a solution of the regular-singular Picard–Fuchs equation

$$\frac{d}{ds} I(s) = P(s) I(s), \qquad \text{with} \quad P(s)_{ij} = p_{ij}(s). \tag{5.4}$$

Similarly, suppose given a holomorphic n-form ω and let ω/df be the corresponding Gelfand–Leray form, defining a section $[\omega/df]$ of the vanishing cohomology bundle. Let $\delta_1, \dots, \delta_\mu$ be a basis of the vanishing homology, $\delta_i(s) \in H_{n-1}(F_s, \mathbb{Z})$. Then the function

$$I(s) = \left(\int_{\delta_1(s)} \frac{\omega}{df}, \dots, \int_{\delta_\mu(s)} \frac{\omega}{df} \right) \tag{5.5}$$

satisfies a regular singular order ℓ differential equation

$$I^{(\ell)}(s) + p_1(s) I^{(\ell-1)}(s) + \cdots + p_\ell(s) I(s) = 0, \tag{5.6}$$

where the order is bounded above by the multiplicity of the critical point (see [5], §12.2.1). One refers to (5.6), or to the equivalent system of regular singular homogeneous first order equations

$$\frac{d}{ds} \mathcal{I}(s) = \mathcal{P}(s) \mathcal{I}(s), \tag{5.7}$$

with

$$\mathcal{I}_r(s) = s^{r-1} I^{(r-1)}(s), \tag{5.8}$$

as the Picard–Fuchs equation of ω. For the relation between Picard–Fuchs equations and mixed Hodge structures see §12 of [5] and [33].

5.2. **Flat equisingular connections.** We first recall some properties of the flat equisingular connections introduced in [22] (see also §1 of [23]). We denote by G the affine group scheme dual to the commutative Hopf algebra of Feynman diagrams, graded by loop number. We let $\mathfrak{g}$ denote the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$. Let K denote the field of germs of meromorphic functions at $z = 0$. We also let B denote a fibration over an infinitesimal disk Δ^* with fiber the multiplicative group $\mathbb{G}_m$ and we denote by P the principal G-bundle $P = B \times G$. We consider $\mathrm{Lie}(G)$-valued flat connections ω that are *equisingular*, *i.e.* they satisfy

- The connections satisfies $\omega(z,\lambda u)=\lambda^Y\omega(z,u)$, for $\lambda\in\mathbb{G}_m$, with Y the grading operator.
- Solutions of $D\gamma=\omega$, have the property that their pullbacks $\sigma^*(\gamma)\in G(K)$ along any section $\sigma:\Delta\to B$ with fixed value $\sigma(0)$ have the same negative piece of the Birkhoff factorization $\sigma^*(\gamma)_-$.

The first condition and the flatness condition imply that the connection $\omega(z,u)$ can be written in the form

$$\omega(z,u)=u^Y(a(z))\,dz+u^Y(b(z))\,\frac{du}{u}, \tag{5.9}$$

where $a(z)$ and $b(z)$ are elements of $\mathfrak{g}(K)$ satisfying the flatness condition

$$\frac{db}{dz}-Y(a)+[a,b]=0. \tag{5.10}$$

Recall that the Lie bracket in the Lie algebra $\mathrm{Lie}(G)$ is obtained by assigning

$$[\Gamma,\Gamma']=\sum_{v\in V(\Gamma)}\Gamma\circ_v\Gamma'-\sum_{v'\in V(\Gamma')}\Gamma'\circ_{v'}\Gamma, \tag{5.11}$$

where $\Gamma\circ_v\Gamma'$ denotes the graph obtained by inserting Γ' into Γ at the vertex $v\in V(\Gamma)$ and the sum is over all vertices where an insertion is possible.

The equisingularity condition, which determines the behavior of pullbacks of solutions along sections of the fibration $\mathbb{G}_m\to B\to\Delta$, can be checked by writing the equation $Df=\omega$ in the more explicit form

$$\gamma^{-1}\frac{d\gamma}{dz}=a(z),\quad\text{and}\quad\gamma^{-1}Y(\gamma)=b(z). \tag{5.12}$$

When one interprets elements $\gamma\in G(K)$ as algebra homomorphisms $\phi\in\mathrm{Hom}(\mathcal{H},K)$, one can write the above equivalently in the form

$$(\phi\circ S)*\frac{d\phi}{dz}=a,\quad\text{and}\quad(\phi\circ S)*Y(\phi)=b, \tag{5.13}$$

where S is the antipode in $\mathcal{H}$ and $*$ is the product dual to the coproduct in the Hopf algebra. This means, on generators Γ of $\mathcal{H}$,

$$\langle(\phi\circ S)\otimes\frac{d\phi}{dz},\Delta(\Gamma)\rangle=a_\Gamma,\quad\text{and}\quad\langle(\phi\circ S)\otimes Y(\phi),\Delta(\Gamma)\rangle=b_\Gamma, \tag{5.14}$$

where

$$\Delta(\Gamma)=\Gamma\otimes 1+1\otimes\Gamma+\sum_\gamma\gamma\otimes\Gamma/\gamma$$

as in (4.23), with the sum over subdivergences, and the antipode is given inductively by

$$S(X)=-X-\sum S(X')X'', \tag{5.15}$$

for $\Delta(X)=X\otimes 1+1\otimes X+\sum X'\otimes X''$, with X' and X'' of lower degree.

5.3. **From regular to irregular singularities.** We now show how to produce a flat connection of the desired form (5.9), with irregular singularities, starting from the graph hypersuraces X_Γ, a consistent choice of slicing Π_ξ, and the regular singular Picard–Fuchs equation associated to the resulting isolated singularities of $X_\Gamma\cap\Pi_\xi$.

We begin by introducing a small modification of the Hopf algebra and coproduct, which accounts for the fact of having to choose a slicing Π_ξ. This is similar to what happens when one enriches the discrete Hopf algebra by adding the data of the external momenta.

Let $\mathcal{S}_\Gamma$ denote the manifold of planes Π_ξ in $\mathbb{A}^{\#E(\Gamma)}$ with $\dim \Pi_\xi \leq \operatorname{codim} \operatorname{Sing}(X_\Gamma)$. We can write $\mathcal{S}_\Gamma$ as a disjoint union

$$\mathcal{S}_\Gamma = \bigcup_{m=1}^{\operatorname{codim} \operatorname{Sing}(X_\Gamma)} \mathcal{S}_{\Gamma,m}, \tag{5.16}$$

where $\mathcal{S}_{\Gamma,m}$ is the manifold of m-dimensional planes in $\mathbb{A}^{\#E(\Gamma)}$. We denote by $\mathcal{C}^\infty(\mathcal{S}_\Gamma)$ the space of test functions on $\mathcal{S}_\Gamma$ and by $\mathcal{C}_c^{-\infty}(\mathcal{S}_\Gamma)$ its dual space of distributions.

Lemma 5.1. *Suppose given a subgraph $\gamma \subset \Gamma$. Then the choice of a distribution $\sigma \in \mathcal{C}_c^{-\infty}(\mathcal{S}_\Gamma)$ induces distributions $\sigma_\gamma \in \mathcal{C}_c^{-\infty}(\mathcal{S}_\gamma)$ and $\sigma_{\Gamma/\gamma} \in \mathcal{C}_c^{-\infty}(\mathcal{S}_{\Gamma/\gamma})$.*

Proof. Given $\gamma \subset \Gamma$, neglecting external edges, we can realize the affine X_γ as a hypersurface inside a linear subspace $\mathbb{A}^{\#E(\gamma)} \subset \mathbb{A}^{\#E(\Gamma)}$ and similarly for the affine $X_{\Gamma/\gamma}$, seen as a hypersurface inside a linear subspace $\mathbb{A}^{\#E(\Gamma/\gamma)} \subset \mathbb{A}^{\#E(\Gamma)}$, where we simply identify the edges of γ or Γ/γ with a subset of the edges of the original graph Γ.

One then has a restriction map $T_\gamma : \mathcal{S}_{\Gamma,\gamma} \to \mathcal{S}_\gamma$, where $\mathcal{S}_{\Gamma,\gamma} \subset \mathcal{S}_\Gamma$ is the union of the components $\mathcal{S}_{\Gamma,m}$ of $\mathcal{S}_\Gamma$ with $m \leq \operatorname{codim} \operatorname{Sing}(X_\gamma)$,

$$\mathcal{S}_{\Gamma,\gamma} = \bigcup_{m=1}^{\operatorname{codim} \operatorname{Sing}(X_\gamma)} \mathcal{S}_{\Gamma,m}, \tag{5.17}$$

which is given by

$$T_\gamma(\Pi_\xi) = \Pi_\xi \cap \mathbb{A}^{\#E(\gamma)}. \tag{5.18}$$

This induces a map $T_\gamma : \mathcal{C}^\infty(\mathcal{S}_\gamma) \to \mathcal{C}^\infty(\mathcal{S}_\Gamma)$ given by

$$T_\gamma(f)(\Pi_\xi) = \begin{cases} f(T_\gamma(\Pi_\xi)) & \Pi_\xi \in \mathcal{S}_{\Gamma,\gamma} \\ 0 & \text{otherwise.} \end{cases} \tag{5.19}$$

In turn, this defines a map $T_\gamma : \mathcal{C}_c^{-\infty}(\mathcal{S}_\Gamma) \to \mathcal{C}_c^{-\infty}(\mathcal{S}_\gamma)$, at the level of distributions, by

$$T_\gamma(\sigma)(f) = \sigma(T_\gamma(f)). \tag{5.20}$$

The argument for Γ/γ is analogous. One sets $\sigma_\gamma = T_\gamma(\sigma_\Gamma)$ and $\sigma_{\Gamma/\gamma} = T_{\Gamma/\gamma}(\sigma_\Gamma)$. $\square$

We then enrich the original Hopf algebra $\mathcal{H}$ by adding the datum of the slicing Π_ξ. We consider the commutative algebra

$$\tilde{\mathcal{H}} = \operatorname{Sym}(\mathcal{C}_c^{-\infty}(\mathcal{S})), \tag{5.21}$$

where $\mathcal{S} = \cup_\Gamma \mathcal{S}_\Gamma$, endowed with the coproduct

$$\Delta(\Gamma,\sigma) = (\Gamma,\sigma) \otimes 1 + 1 \otimes (\Gamma,\sigma) + \sum_\gamma (\gamma,\sigma_\gamma) \otimes (\Gamma/\gamma, \sigma_{\Gamma/\gamma}). \tag{5.22}$$

Lemma 5.2. *The coproduct* (5.22) *is coassociative and $\tilde{\mathcal{H}}$ is a Hopf algebra.*

Proof. The proof is analogous to the one given in [23], Theorem 1.27. $\square$

We then proceed as follows. We pass to the projective instead of affine formulation and we fix a small neighborhood of an isolated singular point of $X_\Gamma \cap \Pi_\xi$, for Π_ξ a linear space of dimension at most equal to the codimension of $\operatorname{Sing}(X_\Gamma)$. Suppose given a holomorphic k-form α_ξ on Π_ξ. Then there exists an associated regular singular Picard–Fuchs equation

$$J_{\Gamma,\xi}^{(\ell)}(s) + p_1(s) J_{\Gamma,\xi}^{(\ell-1)}(s) + \cdots + p_\ell(s) J_{\Gamma,\xi}(s) = 0, \tag{5.23}$$

with the property that any solution $J_{\Gamma,\xi}(s)$ is a linear combination of the functions

$$J_{\Gamma,\xi,i}(s) = \int_{\delta_i(s)} \frac{\alpha_\xi}{df}, \tag{5.24}$$

where $\delta_1, \ldots, \delta_\mu$ be a basis of locally constant sections of the homological Milnor fibration, $\delta_i(s) \in H_{k-1}(F_s, \mathbb{Z})$, and α_ξ/df is the Gelfand–Leray form associated to the holomorphic k-form α_ξ.

This depends on the choice of a singular point and can be localized in a small neighborhood of the singular point in $X_\Gamma \cap \Pi_\xi$. In fact, introducing a cutoff χ_ξ as in (4.33) that is supported near the singularities of $X_\Gamma \cap \Pi_\xi$ amounts to adding the expressions (5.24) for the different singular points. Thus, to simplify notations, we can just assume to have a single expression $J_{\Gamma,\xi}(s)$ at a unique isolated critical point.

We then have the following result, which constructs irregular singular connections as in §5.2 from solutions of the regular singular Picard–Fuchs equation.

Theorem 5.3. *Any solution $J_{\Gamma,\xi}$ of the regular singular Picard–Fuchs equation* (5.23) *determines a flat $\mathfrak{g}(K)$-valued connection $\omega(z,u)$ of the form* (5.9)*. Moreover, if the k-form α_ξ is given by $P_\Gamma^\ell \Omega_\xi$ as in* (4.35)*, then the connection is equisingular.*

Proof. We consider the Mellin transform, as in (4.38)

$$\mathcal{F}_{\Gamma,\xi}(z) = \int_0^\infty s^z \, J_{\Gamma,\xi}(s) \, ds. \tag{5.25}$$

As in Corollary 4.6 (see §7 of [5]), the function $\mathcal{F}_{\Gamma,\xi}(z)$ admits an analytic continuation to meromorphic functions with poles at points $z = -(\lambda+1)$ with $\lambda \in \Xi_{\Gamma,\xi}$ a discrete set in $\mathbb{R}$ of points related to the multiplicities of an embedded resolution of the singular point of $X_\Gamma \cap \Pi_\xi$. We look at the function $\mathcal{F}_{\Gamma,\xi}(z)$ in a small neighborhood of a chosen point $z = -D$. It has an expansion as a Laurent series, with a pole at $z = -D$ if $-D \in \Xi_{\Gamma,\xi}$.

After a change of variables on the complex coordinate z, so that we have $z \in \Delta^*$ a small neighborhood of $z = 0$, we define

$$\phi_\mu(\Gamma, \sigma)(z) := \mu^{-z\, b_1(\Gamma)} \sigma \left(\mathcal{F}_{\Gamma,\xi}(-\frac{D+z}{2}) \right), \tag{5.26}$$

where we consider $\mathcal{F}_{\Gamma,\xi}$ as a function of ξ to which we apply the distribution σ. More precisely, after identifying $\mathcal{F}_{\Gamma,\xi}$ with its Laurent series expansion, we apply σ to the coefficients seen as functions of ξ. This defines an algebra homomorphism $\phi_\mu \in \mathrm{Hom}(\tilde{\mathcal{H}}, K)$, by assigning the values (5.26) on generators. Here μ is the mass scale as in §2.3 above. The homomorphism ϕ defined by (5.26) can be equivalently described as a family of $\tilde{G}(\mathbb{C})$-valued loops $\gamma_\mu : \Delta^* \to \tilde{G}(\mathbb{C})$, depending on the mass scale μ. Here $\tilde{G}$ denotes the affine group scheme dual to the commutative Hopf algebra $\tilde{\mathcal{H}}$. The dependence on μ of (5.26) implies that γ_μ satisfies the scaling property

$$\gamma_{e^t \mu}(z) = \theta_{tz}(\gamma_\mu(z)), \tag{5.27}$$

where θ_t is the one-parameter family of automorphisms of $\tilde{\mathcal{H}}$ generated by the grading, $\frac{d}{dt}\theta_t|_{t=0} = Y$. Then one sets

$$a_\mu(z) := (\phi_\mu \circ S) * \frac{d}{dz}\phi_\mu, \quad \text{and} \quad b_\mu(z) := (\phi_\mu \circ S) * Y(\phi_\mu), \tag{5.28}$$

where S and $*$ are the antipode of $\tilde{\mathcal{H}}$ and the product dual to the coproduct Δ of (5.22). These define elements a_μ, b_μ $\Omega^1(\mathfrak{g}(K))$, which one can use to define a connection $\omega(z,u)$ of the form (5.9). More precisely, for $\mu = e^t$, one has

$$\gamma_\mu^{-1} \frac{d}{dz} \gamma_\mu = \theta_t(\gamma^{-1} \frac{d}{dz} \gamma) = u^Y(a(z)),$$

where we set $u^Y = e^{tY}$ and then extend the resulting expression to $u \in \mathbb{G}_m(\mathbb{C}) = \mathbb{C}^*$. Similarly, we get $\gamma_\mu^{-1} Y(\gamma_\mu) = u^Y(b(z))$. Thus, the connection $\omega(z,u)$ defined in this way satisfies by construction the first condition of the equisingularity property, namely $\omega(z, \lambda u) = \lambda^Y \omega(z,u)$, for all $\lambda \in \mathbb{G}_m$. One can see that the connection is flat since we have

$$\frac{d}{dz} b_\mu(z) - Y(a_\mu(z)) = \frac{d\gamma_\mu^{-1}(z)}{dz} Y(\gamma_\mu(z)) + \gamma_\mu^{-1}(z) \frac{d}{dz}(Y(\gamma_\mu(z)))$$
$$-Y(\gamma_\mu^{-1}(z)) \frac{d}{dz}\gamma_\mu(z) - \gamma_\mu^{-1}(z)\frac{d}{dz}(Y(\gamma_\mu(z)))$$
$$= -\gamma_\mu^{-1}(z)\frac{d}{dz}(\gamma_\mu(z))\gamma_\mu^{-1}(z) Y(\gamma_\mu(z)) - \gamma_\mu^{-1}(z) Y(\gamma_\mu(z))\gamma_\mu^{-1}(z) = -[a(z), b(z)].$$

The second condition of equisingularity is the property that, in the Birkhoff factorization

$$\gamma_\mu(z) = \gamma_{\mu,-}(z)^{-1}\gamma_{\mu,+}(z),$$

the negative part satisfies

$$\frac{d}{d\mu}\gamma_{\mu,-}(z) = 0.$$

By dimensional analysis on the counterterms, in the case of Dimensional Regularization and Minimal Subtraction, it is possible to show (see [20] §5.8.1) that the counterterms obtained by the BPHZ procedure applied to the Feynman integral $U_\mu(\Gamma)(z)$ of (2.49) and (2.48) do not depend on the mass parameter μ. This means, as shown in [21] (see also Proposition 1.44 of [23]), that the Feynman integrals $U_\mu(\Gamma)(z)$ define a $G(\mathbb{C})$-valued loop $\gamma_\mu(z)$ with the property that $\partial_\mu \gamma_{\mu,-}(z) = 0$. The integrals (5.25) considered here, in the case where α_ξ is of the form (4.35), correspond to slices along a linear space Π_ξ of the Feynman integrals (2.49), localized by a cutoff χ_ξ near the singular points. The explicit dependence on μ in the integrals (3.31) is as in (5.26), which is unchanged with respect to that of the original dimensionally regularized Feynman integrals (2.49). Thus, the same argument of [20] §5.8.1 and Proposition 1.44 of [23] applies to this case to show that $\partial_\mu \gamma_{\mu,-}(z) = 0$. □

6. Logarithmic motives, Dimensional Regularization, and motivic sheaves

In this section we propose a candidate for a motivic formulation of dimensional regularization. As we discussed already in §2.2 above, in physics dimensional regularization is intended as a purely formal recipe that assigns a meaning to Gaussian integrals in "complexified dimension" $z \in \mathbb{C}$ by continuation to non-integer values of the usual formula for integer dimensions

$$\int e^{-\lambda t^2} d^z t := \pi^{z/2} \lambda^{-z/2}. \tag{6.1}$$

Usually, in so doing, one does not attempt to give a geometric meaning to the space of integration as a "space in complexified dimension $z \in \mathbb{C}$". The question of whether one can actually make sense of a geometry in complexified dimension was considered in [23], from the point of view of noncommutative geometry, where the usual notion of dimension of a space is replaced by the dimension spectrum, which is a set of complex numbers. A geometric model for a space whose dimension spectrum consists of a single point $z \in \mathbb{C}^*$ is given in §I.19.2 of [23], and it is shown that the formula (6.1) can be recovered from the properties of the Dirac operator on this space.

Here we also consider the question of giving geometric meaning to the complexified dimension, but we try to construct a geometric model underlying the operation of dimensional regularization using motives. We propose a candidate for a motive describing the dimensional regularization of a given Feynman graph. This is defined as an extension

(in fact as a pro-motive) in the category of mixed motives, which is obtained from the logarithmic extension of Tate motives and the motive of the graph hypersurface. We work in the geometric setting of motivic sheaves. One can choose to work in a similar way at the level of Hodge structures, using Hodge sheaves. For a formulation of some aspects of dimensional regularization in a setting that is closer to that of mixed Hodge modules, we refer the reader to §7 of [37] and the unpublished [24].

Just as in the case of noncommutative geometry, where the operation of dimensional regularization is understood as a product of the ordinary space in integer dimension by the "space of dimension z", here we also find that the dimensionally regularized Feynman integral is recovered by taking the product, in a category of motivic sheaves, of the motive associated to the graph hypersurface of a given Feynman graph by this pro-motive representing the "space of dimension z". It would be interesting to find a more explicit relation between this motivic description of dimensional regularization and the one based on noncommutative geometry, described in [23] and [24].

6.1. **Mixed Tate motives and the logarithmic extensions.** We recall briefly the definition of the logarithmic motives, as given in [6]. Let $\mathcal{DM}(\mathbb{G}_m)$ be the Voevodsky category of mixed motives (motivic sheaves) over the multiplicative group $\mathbb{G}_m$. We will assume that the base field $\mathbb{K}$ is a number field (in fact, we can work over $\mathbb{Q}$) so that the extensions considered here take place in an abelian category of mixed Tate motives (*cf.* [2], [35]). Recall that the extensions $\mathrm{Ext}^1_{\mathcal{DM}(\mathbb{K})}(\mathbb{Q}(0),\mathbb{Q}(1))$ of Tate motives are given by the Kummer motives $M=[\mathbb{Z}\stackrel{u}{\to}\mathbb{G}_m]$ with $u(1)=q\in\mathbb{K}^*$. This extension has period matrix of the form

$$\begin{pmatrix} 1 & 0 \\ \log q & 2\pi i \end{pmatrix}. \tag{6.2}$$

When, instead of working with motives over the base field $\mathbb{K}$, one works with the relative setting of motivic sheaves over a base scheme S, instead of the Tate motives $\mathbb{Q}(n)$ one considers the Tate sheaves $\mathbb{Q}_S(n)$. These correspond to the constant sheaf with the motive $\mathbb{Q}(n)$ over each point $s\in S$. In the case where $S=\mathbb{G}_m$, there is a natural way to assemble the Kummer motives into a unique extension in $\mathrm{Ext}^1_{\mathcal{DM}(\mathbb{G}_m)}(\mathbb{Q}_{\mathbb{G}_m}(0),\mathbb{Q}_{\mathbb{G}_m}(1))$. This is the Kummer extension

$$\mathbb{Q}_{\mathbb{G}_m}(1)\to\mathcal{K}\to\mathbb{Q}_{\mathbb{G}_m}(0)\to\mathbb{Q}_{\mathbb{G}_m}(1)[1], \tag{6.3}$$

where over the point $s\in\mathbb{G}_m$ one is taking the Kummer extension $M_s=[\mathbb{Z}\stackrel{u}{\to}\mathbb{G}_m]$ with $u(1)=s$. Because of the logarithm function $\log(s)$ that appears in the period matrix for this extension, the Kummer extension (6.3) is also referred to as the *logarithmic motive*. We use the notation $\mathcal{K}=\mathrm{Log}$ as in [6] to refer to this extension, *cf.* [7].

When working with $\mathbb{Q}$-coefficients, so that one can include denominators in the definition of projectors, one can then consider the logarithmic motives Log^n, defined as in [6] by setting

$$\mathrm{Log}^n=\mathrm{Sym}^n(\mathcal{K}), \tag{6.4}$$

where the symmetric powers of an object in $\mathcal{DM}_{\mathbb{Q}}(\mathbb{G}_m)$ are defined as

$$\mathrm{Sym}^n(X)=\frac{1}{\#\Sigma_n}\sum_{\sigma\in\Sigma_n}\sigma(X^n). \tag{6.5}$$

Recall that the polylogarithms appear naturally as period matrices for extensions involving the symmetric powers $\mathrm{Log}^n=\mathrm{Sym}^n(\mathcal{K})$, in the form [12]

$$0\to\mathrm{Log}^{n-1}(1)\to\mathcal{L}^n\to\mathbb{Q}(0)\to 0, \tag{6.6}$$

where $M(1) = M \otimes \mathbb{Q}(1)$ and $\mathcal{L}^1 = \mathrm{Log}$. The mixed motive $\mathcal{L}^n$ has period matrix

$$\begin{pmatrix} 1 & 0 \\ M_{\mathrm{Li}}^{(n)} & M_{\mathrm{Log}^{n-1}(1)} \end{pmatrix} \tag{6.7}$$

with

$$M_{\mathrm{Li}}^{(n)} = (-\mathrm{Li}_1(s), -\mathrm{Li}_2(s), \cdots, -\mathrm{Li}_n(s))^\tau, \tag{6.8}$$

where τ means transpose and where

$$\mathrm{Li}_1(s) = -\log(1-s), \quad \text{and} \quad \mathrm{Li}_n(s) = \int_0^s \mathrm{Li}_{n-1}(u)\,\frac{du}{u},$$

equivalently defined (on the principal branch) using the power series

$$\mathrm{Li}_n(s) = \sum_k \frac{s^k}{k^n},$$

and with

$$M_{\mathrm{Log}^n(1)} = \begin{pmatrix} 2\pi i & 0 & 0 & \cdots & 0 \\ 2\pi i \log(s) & (2\pi i)^2 & 0 & \cdots & 0 \\ 2\pi i \frac{\log^2(s)}{2!} & (2\pi i)^2 \log(s) & (2\pi i)^3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 2\pi i \frac{\log^n(s)}{n!} & (2\pi i)^2 \frac{\log^{n-1}(s)}{(n-1)!} & (2\pi i)^3 \frac{\log^{n-2}(s)}{(n-2)!} & \cdots & (2\pi i)^n \end{pmatrix}. \tag{6.9}$$

The period matrices for the motives Log^n correspond to the description of Log^n as extension of $\mathbb{Q}(0)$ by $\mathrm{Log}^{n-1}(1)$, *i.e.* to the distinguished triangles in $\mathcal{DM}(\mathbb{G}_m)$ of the form

$$\mathrm{Log}^{n-1}(1) \to \mathrm{Log}^n \to \mathbb{Q}(0) \to \mathrm{Log}^{n-1}(1)[1]. \tag{6.10}$$

The motives Log^n form a projective system under the canonical maps

$$\beta_n : \mathrm{Log}^{n+1} \to \mathrm{Log}^n$$

given by the composition of the morphisms $\mathrm{Sym}^{n+m}(\mathcal{K}) \to \mathrm{Sym}^n(\mathcal{K}) \otimes \mathrm{Sym}^m(\mathcal{K})$, as in [6], Lemma 4.35, given by the fact that $\mathrm{Sym}^{n+m}(\mathcal{K})$ is canonically a direct factor of $\mathrm{Sym}^n(\mathcal{K}) \otimes \mathrm{Sym}^m(\mathcal{K})$, and the map $\mathrm{Sym}^m(\mathcal{K}) \to \mathbb{Q}(0)$ of (6.10), in the particular case $m = 1$. Let Log^∞ denote the pro-motive obtained as the projective limit

$$\mathrm{Log}^\infty = \varprojlim_n \mathrm{Log}^n. \tag{6.11}$$

The analog of the period matrix (6.9) becomes then the infinite matrix

$$M_{\mathrm{Log}^\infty(1)} = \begin{pmatrix} 2\pi i & 0 & 0 & \cdots & 0 & \cdots \\ 2\pi i \log(s) & (2\pi i)^2 & 0 & \cdots & 0 & \cdots \\ 2\pi i \frac{\log^2(s)}{2!} & (2\pi i)^2 \log(s) & (2\pi i)^3 & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \cdots \\ 2\pi i \frac{\log^n(s)}{n!} & (2\pi i)^2 \frac{\log^{n-1}(s)}{(n-1)!} & (2\pi i)^3 \frac{\log^{n-2}(s)}{(n-2)!} & \cdots & (2\pi i)^n & \cdots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \cdots \end{pmatrix}. \tag{6.12}$$

In other words, the mixed Hodge structure associated to the motives Log^n is the one that has as the weight filtrations W_{-2k} the range of multiplication by the matrix M_{Log^n} defined as in (6.9) on vectors in $\mathbb{Q}^n$ with the first $k-1$ entries equal to zero, while the Hodge filtration F^{-k} is given by the range of multiplication of M_{Log^n} on vectors of $\mathbb{C}^n$ with the entries from $k+1$ to n equal to zero [12].

Thus, in this Hodge realization, the H^0 piece corresponds to the first column of the matrix M_{Log^n}, where the k-th entry corresponds to the k-th graded piece of the weight filtration. Let us consider the corresponding grading operator, that multiplies the k-th entry by T^k. One can then associate to the h^0-piece of the Log^∞ motive the following formal expression that corresponds in the period matrix (6.12) to the H^0 part in the MHS realization:

$$\mathbb{Q}\cdot\sum_k \frac{\log^k(s)}{k!}T^k =: \mathbb{Q}\cdot s^T. \tag{6.13}$$

The formal expression (6.13) has in fact an interpretation in terms of periods. This follows from a well known result (*cf. e.g.* [31], Lemma 2.10) expressing the powers of the logarithm in terms of iterated integrals. For iterated integrals we use the notation as in [31]

$$\int_a^b \frac{ds}{s}\circ\frac{ds}{s}\circ\cdots\frac{ds}{s} = \int_{a\le s_1\le\cdots\le s_n\le b}\frac{ds_1}{s_1}\wedge\cdots\wedge\frac{ds_n}{s_n}. \tag{6.14}$$

We also denote by $\Lambda_{a,b}(n)$ the domain

$$\Lambda_{a,b}(n) = \{(s_1,\ldots,s_n)\,|\,a\le s_1\le\cdots\le s_n\le b\}. \tag{6.15}$$

Lemma 6.1. *The expression* (6.13) *is obtained as rational multiples of the pairing*

$$s^T = \int_{\Lambda_{1,s}(\infty)}\eta(T), \tag{6.16}$$

with $\Lambda_{1,s}(\infty) = \cup_n\Lambda_{1,s}(n)$ *and the form*

$$\eta(T) := \sum_n \frac{ds_1}{s_1}\wedge\cdots\wedge\frac{ds_n}{s_n}\,T^n. \tag{6.17}$$

Proof. The result follows from the basic identity (*cf.* [31], Lemma 2.10)

$$\int_{\Lambda_{a,b}(n)}\frac{ds_1}{s_1}\wedge\cdots\wedge\frac{ds_n}{s_n} = \frac{\log\left(\frac{b}{a}\right)^n}{n!}. \tag{6.18}$$

□

6.2. **Motivic sheaves and graph hypersurfaces.** Arapura constructed in [3] a category of motivic sheaves over a base scheme S, modeled on Nori's approach to the construction of categories of mixed motives. We discuss briefly how a similar formalism may be applied to the Feynman motives associated to the graph hypersurfaces with the corresponding periods of the form (2.19).

The category of motivic sheaves constructed in [3] is based on Nori's construction of categories of motives via representations of graphs made of objects and morphisms (*cf.* [19]). In Arapura's case, one constructs a category of motivic sheaves over a scheme S, by taking as vertices of the corresponding graph objects of the form

$$(f : X\to S, Y, i, w)\,, \tag{6.19}$$

where $f : X\to S$ is a quasi-projective morphism, $Y\subset X$ is a closed subvariety, $i\in\mathbb{N}$, and $w\in\mathbb{Z}$. One thinks of such an object as determining a motivic version $h^i_S(X,Y)(w)$ of the local system given by the (Tate twisted) fiberwise cohomology of the pair $H^i_S(X,Y;\mathbb{Q}) = R^if_*j_!\mathbb{Q}_{X\smallsetminus Y}$, where $j = j_{X\smallsetminus Y} : X\smallsetminus Y\hookrightarrow X$ is the open inclusion, *i.e.* the sheaf defined by

$$U\mapsto H^i(f^{-1}(U), f^{-1}(U)\cap Y;\mathbb{Q}).$$

The edges are given by the geometric morphisms, *i.e.* morphisms of varieties over S,

$$(f_1 : X_1 \to S, Y_1, i, w) \to (f' : X_2 \to S, Y_2 = F(Y), i, w), \quad \text{with} \quad f_2 \circ F = f_1; \tag{6.20}$$

the connecting morphisms

$$(f : X \to S, Y, i+1, w) \to (f|_Y : Y \to S, Z, i, w), \quad \text{for} \quad Z \subset Y \subset X; \tag{6.21}$$

and the twisted projection morphisms

$$(f : X \times \mathbb{P}^1 \to S, Y \times \mathbb{P}^1 \cup X \times \{0\}, i+2, w+1) \to (f : X \to S, Y, i, w). \tag{6.22}$$

The product in the category of motivic sheaves of [3] is given by the fibered product

$$\begin{aligned} &(X \to S, Y, i, w) \times (X' \to S, Y', i', w') = \\ &(X \times_S X' \to S, Y \times_S X' \cup X \times_S Y', i+i', w+w'). \end{aligned} \tag{6.23}$$

This has the following effect on period computations.

Lemma 6.2. *Suppose then given $\Sigma \subset X$ and $\Sigma' \subset X'$, defining relative homology cycles for (X, Y) and (X', Y'), respectively. One then has, for the fibered product* (6.23)*, the period pairing*

$$\int_{\Sigma \times_S \Sigma'} \pi_X^*(\omega) \wedge \pi_{X'}^*(\eta) = \int_\Sigma \omega \wedge f^* f'_*(\eta), \tag{6.24}$$

where $f : \Sigma \to S$ and $f' : \Sigma' \to S$ are the restrictions of the maps $X \to S$ and $X' \to S$.

Proof. First recall that, when integrating a differential form on a fibered product, one has the formula

$$\int_{X \times_S X'} \pi_X^*(\omega) \wedge \pi_{X'}^*(\eta) = \int_X \omega \wedge (\pi_X)_* \pi_{X'}^*(\eta) = \int_X \omega \wedge f^* f'_*(\eta), \tag{6.25}$$

which corresponds to the diagram

$$\begin{tikzcd} & X \times_S X' \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_{X'}"] & \\ X \arrow[dr, "f"'] & & X' \arrow[dl, "f'"] \\ & S & \end{tikzcd} \tag{6.26}$$

Suppose then given $\Sigma \subset X$ such that $\partial \Sigma \subset Y$ and $\Sigma' \subset X'$ with $\partial \Sigma' \subset Y'$. One has

$$\partial(\Sigma \times_S \Sigma') = \partial\Sigma \times_S \Sigma' \cup \Sigma \times_S \partial\Sigma' \subset Y \times_S X' \cup X \times_S Y',$$

so that $\Sigma \times_S \Sigma'$ defines a relative homology class in $(X \times_S X', Y \times_S X' \cup X \times_S Y')$. Given elements $[\omega] \in H_S^\cdot(X, Y)$ and $[\eta] \in H_S^\cdot(X', Y')$, we then apply the formula (6.25) to the integration on $\Sigma \times_S \Sigma'$ and obtain (6.24). $\square$

6.3. **Logarithmic Feynman motives.** Consider then the graph polynomial $\Psi_\Gamma(s) = \det(M_\Gamma(s))$. By removing the set of zeros of Ψ_Γ, *i.e.* the graph hypersurface X_Γ, we can consider Ψ_Γ as a morphism

$$\Psi_\Gamma : \mathbb{A}^{\# E_\Gamma} \smallsetminus \hat{X}_\Gamma \to \mathbb{G}_m. \tag{6.27}$$

We can then consider the pullback of the logarithmic motive $\mathrm{Log} \in \mathcal{DM}(\mathbb{G}_m)$ by this morphism, as in the construction of the logarithmic specialization system given in [6]. This gives a motive

$$\mathrm{Log}_\Gamma := \Psi_\Gamma^*(\mathrm{Log}) \in \mathcal{DM}(U_\Gamma), \tag{6.28}$$

where $U_\Gamma = \mathbb{A}^{\# E_\Gamma} \smallsetminus \hat{X}_\Gamma$.

In fact, a more sophisticated approach would involve considering here the "log complex" as in §9.2 of [36], *cf.* also §9.4 of [36], see also [30].

In the context of the category of motivic sheaves of Arapura recalled above, we can define the Feynman motives as follows.

Definition 6.3. *The category of Feynman motivic sheaves, for a fixed scalar quantum field theory, is the subcategory of the Arapura category of motivic sheaves over $\mathbb{G}_m$ spanned by the objects of the form*

$$(\Psi_\Gamma : \mathbb{A}^{\# E(\Gamma)} \smallsetminus \hat{X}_\Gamma \to \mathbb{G}_m, \Lambda \smallsetminus (\Lambda \cap \hat{X}_\Gamma), \# E(\Gamma) - 1, \# E(\Gamma) - 1), \tag{6.29}$$

where Γ ranges over the Feynman graphs of the given scalar field theory, and where

$$\Lambda = \{ t \in \mathbb{A}^{\# E(\Gamma)} \mid \prod_i t_i = 0 \} \tag{6.30}$$

is the union of the coordinate hyperplanes.

The above correspond to the local systems

$$H^{n-1}_{\mathbb{G}_m}(\mathbb{A}^n \smallsetminus \hat{X}_\Gamma, \Lambda \smallsetminus (\Lambda \cap \hat{X}_\Gamma), \mathbb{Q}(n-1)), \tag{6.31}$$

with $n = \# E_{int}(\Gamma)$.

One can also include as part of the data the slicing by all possible k-dimensional linear spaces $\Pi_\xi \subset \mathbb{A}^{\# E(\Gamma)}$, with $k \leq \operatorname{codim} \operatorname{Sing}(X_\Gamma)$, as we did in our previous discussions, and consider instead of the (6.29) objects of the form

$$(\Psi_\Gamma|_{\Pi_\xi} : \Pi_\xi \smallsetminus (\hat{X}_\Gamma \cap \Pi_\xi) \to \mathbb{G}_m, (\Lambda \cap \Pi_\xi) \smallsetminus (\Lambda \cap \hat{X}_\Gamma \cap \Pi_\xi), k-1, w). \tag{6.32}$$

Remark 6.4. The reason for taking the cohomology (6.31) relative to the *algebraic simplex* Λ, that is, the union of the coordinate hyperplanes defined by (6.30) is that, in this way, we can regard the *topological simplex* $\Sigma = \{ t \in \mathbb{R}^n_+ \mid \sum_{i=1}^n t_i = 1 \}$ as defining a homology cycle, since $\partial \Sigma \subset \Lambda$.

6.4. **Dimensional Regularization and motives.** In these terms, the procedure of dimensional regularization can then be described as follows. Consider again the logarithmic (pro)motive, viewed itself as a motivic sheaf $X_{\mathrm{Log}^\infty} \to \mathbb{G}_m$ over $\mathbb{G}_m$. One can then take the product of a Feynman motive

$$(\Psi_\Gamma : \mathbb{A}^n \smallsetminus \hat{X}_\Gamma \to \mathbb{G}_\mathfrak{m}, \Lambda \smallsetminus (\Lambda \cap \hat{X}_\Gamma), k-1, k-1),$$

or more generally one of the form (6.32), by the (pro)motive

$$(X^\infty_{\mathrm{Log}} \to \mathbb{G}_m, \Lambda_\infty, 0, 0), \tag{6.33}$$

where Λ_∞ is such that the domain of integration $\Lambda_{1,t}(\infty)$ of the period computation of Lemma 6.1 defines a cycle. The product is given by a fibered product as in (6.23), namely

$$\begin{tikzcd} \Psi_\Gamma^*(\mathrm{Log}^\infty) = (\mathbb{A}^n \smallsetminus \hat{X}_\Gamma) \rtimes_{\mathbb{G}_m} X_{\mathrm{Log}^\infty} \arrow[r] \arrow[d] & X_{\mathrm{Log}^\infty} \arrow[d] \\ \mathbb{A}^n \smallsetminus \hat{X}_\Gamma \arrow[r, "\Psi_\Gamma"] & \mathbb{G}_m \end{tikzcd} \tag{6.34}$$

We then have the following interpretation of the dimensionally regularized Feynman integrals.

Proposition 6.5. *The dimensionally regularized Feynman integral $F_{\Gamma,\xi}(z)$ of* (4.34) *are periods on the product, in the category of motivic sheaves enlarges to include projective limits, of the Feynman motive* (6.32) *by the logarithmic pro-motive* Log^∞ *seen as the motivic sheaf* (6.33).

Proof. Consider the product (6.34), with the two projections

$$\pi_X : (\Pi_\xi \smallsetminus (\hat{X}_\Gamma \cap \Pi_\xi)) \times_{\mathbb{G}_m} X_{\mathrm{Log}^\infty} \to \Pi_\xi \smallsetminus (\hat{X}_\Gamma \cap \Pi_\xi)$$
$$\pi_L : (\Pi_\xi \smallsetminus (\hat{X}_\Gamma \cap \Pi_\xi)) \times_{\mathbb{G}_m} X_{\mathrm{Log}^\infty} \to X_{\mathrm{Log}^\infty}.$$

and the form $\pi_X^*(\alpha_\xi) \wedge \pi_L^*(\eta(T))$, where α_ξ is as in (4.35), and $\eta(T)$ is the form on X_{Log^∞} that gives the period (6.16). The period computation of Lemma 6.1 gives

$$\Psi_\Gamma^*\left(\int_{\Lambda_{1,s}(\infty)} \eta(T)\right) = \int_{\Lambda_{1,\Psi_\Gamma(t)}(\infty)} \eta(T) = \sum_n \frac{\log(\Psi_\Gamma(t))^n}{n!} T^n = \Psi_\Gamma(t)^T. \tag{6.35}$$

We then have, by (6.24),

$$\int_{(\Sigma\cap\Pi_\xi)\times_{\mathbb{G}_m}\Lambda_{1,\Psi_\Gamma(t)}(\infty)} \pi_X^*(\alpha_\xi) \wedge \pi_L^*(\eta(T)) = \int_{\Sigma\cap\Pi_\xi} \alpha_\xi \wedge (\pi_X)_*\pi_L^*(\eta(T)) = \int_{\Sigma\cap\Pi_\xi} \Psi_\Gamma^T \alpha_\xi.$$

This is the integral (4.34), up to replacing the formal variable T of (6.13) with the complex DimReg variable z. ☐

The interpretation that emerges from this calculation is that performing the dimensional regularization of a Feynman integral can be thought of as taking the product in the category of motivic sheaves of the motive (motivic sheaf) of the graph hypersurface by the projective limit of the logarithmic motives. The variable $z \in \mathbb{G}_m$ that gives the complexified dimension of dimensional regularization corresponds to the 1-parameter group generated by the grading operator associated to the weight filtration of the logarithmic motives. The dimensionally regularized integral is then a period of this product motive.

6.5. **Motivic zeta function and the DimReg integral.** Kapranov introduced a notion of motivic zeta function by defining

$$Z_X(T) := \sum_{n\geq 0} \mathrm{Sym}^{\mathrm{n}}(X)\, T^n, \tag{6.36}$$

where the $\mathrm{Sym}^n(X)$ can be regarded as objects in an abelian category of motives, or as classes $[\mathrm{Sym}^n(X)]$ in the corresponding Grothendieck ring. Kapranov proved that, when X is the motive of a curve, then the zeta function is a rational function, in the sense that, given a motivic measure $\mu : K_0(\mathcal{M}) \to A$, the zeta function $Z_{X,\mu}(T) \in A[[T]]$ is a rational function of T. Later, Larsen and Lunts showed that in general this is not true in the case of algebraic surfaces [34].

Here we consider the motivic zeta function of the pullback of the logarithmic motive along the function Ψ_Γ as in (6.27). Namely, we consider the motivic zeta function

$$Z_{\mathrm{Log},\Gamma}(T) := \sum_{n\geq 0} \mathrm{Sym}^{\mathrm{n}}(\mathrm{Log}_\Gamma)\, T^n. \tag{6.37}$$

An interesting question, which we do not address in the present paper, is whether one can define a motivic lift of the Dimensional Regularization of the Feynman integral associated to a Feynman graph Γ using the motivic zeta function (6.37). In other words, whether one can obtain the zeta function

$$Z_\Gamma(T) := \sum_{n\geq 0} \frac{\log^n \Psi_\Gamma}{n!} T^n = \Psi_\Gamma^T \tag{6.38}$$

and the associated integrals

$$\sum_{n\geq 0} \left(\int_{\Sigma\cap\Pi_\xi} \frac{\log^n \Psi_\Gamma}{n!} \alpha_\xi \right) T^n = \int_{\Sigma\cap\Pi_\xi} \Psi_\Gamma^T \alpha_\xi \tag{6.39}$$

in a natural way from the motivic zeta function (6.37) of $\Psi_\Gamma^*(\mathrm{Log})$. We hope to return to this and related questions in following work.

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